ON THE AXIOMATIC DEFINITION OF REAL JB*-TRIPLES

ANTONIO M. PERALTA

Abstract. In the last twenty years, a theory of real Jordan triples has been developed. In 1994 T. Dang and B. Russo introduced the concept of J*B-triple. These J*B-triples include real C*-algebras and complex JB*-triples. However, concerning J*B-triples, an important problem was left open. Indeed, the question was whether the complexification of a J*B-triple is a complex JB*-triple in some norm extending the original norm. T. Dang and B. Russo solved this problem for commutative J*B-triples.

In this paper we characterize those J*B-triples with a unitary element whose complexifications are complex JB*-triples in some norm extending the original one. We actually find a necessary and sufficient new axiom to characterize those J*B-triples with a unitary element which are J*B-algebras in the sense of [1] or real JB*-triples in the sense of [4].

1. Introduction

We recall that a real (respectively, complex) *Banach Jordan triple* is a real (respectively, complex) Banach space U with a continuous trilinear (respectively, bilinear in the outer variables and conjugate linear in the middle one) product

$$U \times U \times U \to U$$

$$(xyz) \mapsto \{x, y, z\}$$

satisfying

(1) $\{x, y, z\} = \{z, y, x\}$;

(2) Jordan Identity:

$$L(a, b)\{x, y, z\} - \{L(a, b)x, y, z\} =$$

$$-\{x, L(b, a)y, z\} + \{x, y, L(a, b)z\}$$

for all a, b, x, y, z in U, where $L(x, y)z := \{x, y, z\}$.

2000 *Mathematics Subject Classification.* Primary 17C65, 46L05, and 46L70.

Partially supported by D.G.I.C.Y.T. project no. PB 98-1371, and Junta de Andalucía grant FQM 0199 .

A Banach Jordan triple U is *commutative* or *abelian* if

$$\{\{x,y,z\},u,v\} = \{x,y,\{z,u,v\}\} = \{x,\{y,z,u\},v\}$$

for all $x,y,z,u,v \in U$. An element $u \in U$ is said to be unitary if $L(u,u)$ coincides with the identity map on U.

A complex Jordan Banach triple $\mathcal{E}$ is said to be a *(complex) JB*-triple* if

- (a) The map $L(a,a)$ from $\mathcal{E}$ to $\mathcal{E}$ is an hermitian operator with non negative spectrum for all a in $\mathcal{E}$;
- (b) $\|\{a,a,a\}\| = \|a\|^3$ for all a in $\mathcal{E}$.

We recall that a bounded linear operator T on a complex Banach space is said to be *hermitian* if $\|\exp(i\alpha T)\| = 1$ for all real α.

Complex JB*-triples were introduced by W. Kaup in the study of Bounded Symmetric Domains in complex Banach spaces (see [5], [6]).

During the 80's and early 90's three notions of JB*-triples over the real field were introduced by H. Upmeier [9], T. Dang and B. Russo [3] and J. M. Isidro, W. Kaup and A. Rodríguez [4], respectively.

In 1994 T. Dang and B. Russo [3, Definition 1.3] gave the following axiomatic definition of a JB*-triple over the real field. A *J*B-triple* is a real Banach space E equipped with a structure of a real Banach Jordan triple which satisfies

(J*B 1) $\|\{x,x,x\}\| = \|x\|^3$ for all x in E;
(J*B 2) $\|\{x,y,z\}\| \leq \|x\|\,\|y\|\,\|z\|$ for all x,y,z in E;
(J*B 3) $\sigma^{\mathbb{C}}_{BL(E)}(L(x,x)) \subset [0,+\infty)$ for all $x \in E$;
(J*B 4) $\sigma^{\mathbb{C}}_{BL(E)}(L(x,y) - L(y,x)) \subset i\mathbb{R}$ for all $x,y \in E$.

Every closed subtriple of a J*B-triple is a J*B-triple (c.f. [3, Remark 1.5]). The class of J*B-triples includes real C*-algebras and complex JB*-triples. Moreover, in [3, Proposition 1.4] it is shown that complex JB*-triples are precisely those complex Banach Jordan triples whose underlying real Banach space is a J*B-triple. In [3] the following important problems concerning J*B-triples were left open

- $(P1)$ Is the complexification of a J*B-triple a complex JB*-triple in some norm extending the original norm?
- $(P2)$ Is the bidual of a J*B-triple a J*B-triple with a separately weak*-continuous triple product?

Problem $(P1)$ was affirmatively solved by T. Dang and B. Russo in the particular case of a commutative J*B-triple [3, Theorem 3.11]. They proved that the complexification of every commutative J*B-triple E is a complex JB*-triple in some norm extending the norm on E.

In this paper we study both problems in the general case of J*B-triples with a unitary element. We characterize those J*B-triples having a unitary element whose complexifications are complex JB*-triples in some norm extending the original norm. As a consequence we solve $(P2)$ for the class of J*B-triples with a unitary element where problem $(P1)$ has an affirmative solution.

In 1997 real JB*-triples were introduced by J. M. Isidro, W. Kaup, and A. Rodríguez [4], as norm-closed real subtriples of complex JB*-triples. Isidro, Kaup, and Rodríguez [4, Proposition 2.2] proved that given a real JB*-triple E, then there exists a unique complex JB*-triple structure on its complexification with a norm extending the norm on E.

Let X be a complex Banach space. A *conjugation* on X is a conjugate-linear isometry on X of period 2. If τ is a conjugation on X, then X^τ will stand for the real Banach space of all τ-fixed elements of X. Real normed spaces which can be written as X^τ, for some conjugation τ on X, are called *real forms* of X.

If $\mathcal{E}$ is a complex JB*-triple, then conjugations on $\mathcal{E}$ preserve the triple product of E (c.f. [6, Proposition 5.5]), and hence the real forms of $\mathcal{E}$ are real JB*-triples. In [4] it is shown that actually every real JB*-triple can be regarded as a real form of a suitable complex JB*-triple.

The class of real JB*-triples includes all JB-algebras, all real C*-algebras, all J*B-algebras (compare [1]), and obviously all complex JB*-triples. In [4, Proposition 2.5] it is shown that every real JB*-triple is a J*B-triple but the converse is an open problem. Problem:

$(P1')$ Is every J*B-triple a real JB*-triple?

Problem $(P1')$ has been also stated in [4, page 318]. Actually problems $(P1)$ and $(P1')$ are equivalent.

In this paper (see Theorem 2.5) we also find a necessary and sufficient fifth axiom for a J*B-triple E with a unitary element to be a real JB*-triple in the sense of [4]. This new axiom assures that the operator $L(x, x)$ has non negative numerical range (see definition below). Actually, this new axiom characterizes those J*B-triples with a unitary element which are J*B-algebras in the sense of [1].

2. The results

Let X be a Banach space. Through the paper we denote by B_X, S_X, and X^* the closed unit ball, the unit sphere, and the dual space, respectively, of X. I_X will denote the identity operator on X, and if Y is another Banach space, then $BL(X, Y)$ stands for the Banach space of all bounded linear operators from X to Y. We usually write $BL(X)$ instead of $BL(X,X)$.

Let E be a J*B-triple with a unitary element u. Then E is a unital real Jordan Banach algebra with product $x \circ y := \{x,u,y\}$ and involution $x^* := \{u,x,u\}$ (compare [9, Proposition 19.13]). By (J*B 2) and (J*B 1) it follows that $\|x^*\| = \|x\|$ and $\|x\|^3 = \|U_x(x^*)\|$ for all $x \in E$, where $U_x(y) := 2x \circ (x \circ y) - x^2 \circ y$. Throughout the paper we will denote by E_{sym} the set of all $*$-symmetric elements in E.

Let X be a Banach space, and u a norm-one element in X. The set of states of X relative to u, $D(X,u)$, is defined as the non empty, convex, and weak*-compact subset of X^* given by

$$D(X,u) := \{\Phi \in B_{X^*} : \Phi(u) = 1\}.$$

For $x \in X$, the *numerical range* of x relative to u, $V(X,u,x)$, is given by $V(X,u,x) := \{\Phi(x) : \Phi \in D(X,u)\}$. The *numerical radius* of x relative to u, $v(X,u,x)$, is given by

$$v(X,u,x) := \max\{|\lambda| : \lambda \in V(X,u,x)\}.$$

It is well known that a bounded linear operator T on a complex Banach space X is hermitian if and only if $V(BL(X), I_X, T) \subseteq \mathbb{R}$ (compare [2, Corollary 10.13]).

It is well known that for an element a of a unital Banach algebra A with unit u the following assertions hold

- (a) $V(A,u,a) = V(BL(A), I_A, L_a) = V(BL(A), I_A, R_a)$, where L_a (respectively, R_a) is the map given by $x \mapsto ax$ (respectively $x \mapsto xa$);
- (b) If B is a subalgebra of A with $\{a,u\} \subset B$, then $V(B,u,a) = V(A,u,a)$.

Remark 2.1. *Let X be a complex Banach space and τ a conjugation on X. We define a conjugation $\widetilde{\tau}$ on $BL(X)$ by $\widetilde{\tau}(T) := \tau T \tau$. If T is a $\widetilde{\tau}$-invariant element of $BL(X)$, then we have $T(X^\tau) \subseteq X^\tau$, and hence we can consider $\Lambda(T) := T|_{X^\tau}$ as a bounded linear operator on the real Banach space X^τ. Since the mapping $\Lambda : BL(X)^{\widetilde{\tau}} \to BL(X^\tau)$ is a linear contraction sending I_X to I_{X^τ}, we get*

$$V(BL(X^\tau), I_{X^\tau}, \Lambda(T)) \subseteq V(BL(X)^{\widetilde{\tau}}, I_X, T)$$

for all $T \in BL(X)^{\widetilde{\tau}}$. On the other hand, by the Hahn-Banach Theorem, we have

$$V(BL(X)^{\widetilde{\tau}}, I_X, T) = V(BL(X)_{\mathbb{R}}, I_X, T)$$

for every $T \in BL(X)^{\widetilde{\tau}}$. It follows

$$V(BL(X^\tau), I_{X^\tau}, \Lambda(T)) \subseteq \Re e\ V(BL(X), I_X, T)$$

for all $T \in BL(X)^{\tilde{\tau}}$.

By [8, Lemma 1.6] *we know that if we denote by* $\mathcal{H}$ *the real Banach space of all hermitian operators on* X *which lie in* $BL(X)^{\tilde{\tau}}$, *then for every* $\Phi \in D(BL(X), I_X)$, *there exists* $\Psi \in D(BL(X^\tau, I_{X^\tau})$ *such that* $\Phi(T) = \Psi(\Lambda(T))$ *for every* T *in* $\mathcal{H}$. *As a consequence*

$$V(BL(X), I_X, T) = V(BL(X^\tau), I_{X^\tau}, \Lambda(T))$$

for every $T \in \mathcal{H}$.

Let E be a J*B-triple with a unitary element. The next proposition allows us to calculate the norm of a symmetric element $x \in E$ from the numerical radius of the map $L(x, x)$.

Proposition 2.2. *Let* E *be a J*B-triple with a unitary element* u. *Then for every* $x \in E_{sym}$ *the following assertions hold*

$$V(BL(E), I_E, L(x, x)) \subset [0, +\infty),$$

and

$$\|x\|^2 = \|x^2\| = v(E, u, x^2) = v(BL(E), I_E, L(x, x)).$$

Moreover

$$\|2x \circ x^*\| = v(E, u, L_{2x\circ x^*}) = v(BL(E), I_E, L(x, x) + L(x^*, x^*))$$

for all $x \in E$.

Proof. Let $x \in E_{sym}$. We denote by $C(x) := \overline{Span}^{\|\cdot\|}\{x^n : n = 0, 1, 2, \ldots\}$ the closed subalgebra of E generated by x and u. Since $x^* = x$ we conclude that $C(x)$ is a commutative closed subtriple of E. By [3, Remark 1.5], it follows that $C(x)$ is a commutative J*B-triple with a unitary element. Now Theorem 3.11 in [3] assures us that there exists a commutative complex JB*-triple $B(x)$ and a conjugation τ on $B(x)$ such that $C(x) = B(x)^\tau$. Since $B(x)$ is a commutative complex JB*-triple with a unitary element u, we can conclude that $B(x)$ is a commutative C*-algebra with product $z \circ y = \{z, u, y\}$ and involution $z^* = \{u, z, u\}$. Moreover, the conjugation τ on $B(x)$ satisfies $\tau(z^*) = (\tau(z))^*$ for all $z \in B(x)$. By Remark 2.1, the identity $L(x, x) = L_{x^2}$, and since $C(x)$ is a subalgebra of E containing $\{x, u\}$, it follows that

$$\begin{aligned} V(BL(B(x)), I_{B(x)}, L(x, x)) &= V(BL(C(x)), I_{C(x)}, L(x, x)) \\ &= V(BL(C(x)), I_{C(x)}, L_{x^2}) = V(C(x), u, x^2) = V(E, u, x^2) \\ &= V(BL(E), I_E, L(x, x)). \qquad (1) \end{aligned}$$

Since $B(x)$ is a complex JB*-triple, and hence $L(x,x)$ is hermitian and has non-negative spectrum, it follows from (1) and [2, Lemma 38.3] that

$$V(BL(E), I_E, L(x,x)) = V(BL(B(x)), I_{B(x)}, L(x,x)) \subset [0, +\infty).$$

Since $B(x)$ is a commutative C*-algebra and $x \in B(x)$ is an hermitian element ($x^* = \{u,x,u\} = x$), Sinclair's theorem (c.f. [2, Theorem 10.17]) and equality (1) assure that

$$\|L(x,x)\| = \|x\|^2 = \|x^2\| = v(B(x), u, x^2)$$

$$= v(BL(B(x)), I_{B(x)}, L_{x^2}) = v(BL(B(x)), I_{B(x)}, L(x,x))$$

$$= v(BL(E), I_E, L(x,x)) = v(E, u, x^2).$$

Finally, let $x \in E$. From the equality $L(x,x) + L(x^*, x^*) = L_{2x\circ x^*}$ and following the arguments given above, we conclude that

$$\|2x\circ x^*\| = v(B(x\circ x^*), u, 2x\circ x^*) = v(BL(B(x\circ x^*)), I_{B(x\circ x^*)}, L_{2x\circ x^*})$$

$$= v(BL(C(x\circ x^*)), I_{C(x\circ x^*)}, L_{2x\circ x^*}) = v(C(x\circ x^*)), u, 2x\circ x^*)$$

$$= v(E, u, 2x\circ x^*) = v(BL(E), I_E, L(x,x) + L(x^*, x^*)).$$

□

The following corollary shows that the symmetric part of a J*B-triple with a unitary element is a JB-algebra.

Corollary 2.3. *Let E be a J*B-triple with a unitary element u. Then E_{sym} is a JB-algebra.*

Proof. By Proposition 2.2 the equality

$$\|x\|^2 = v(BL(E), I_E, L(x,x)) = v(E, u, x^2) \tag{2}$$

holds for all $x \in E_{sym}$.

Now let $x, y \in E_{sym}$ and $\phi \in S_{E^*}$ with $\phi(u) = 1$. By Proposition 2.2, we also have $V(E, u, z^2) = V(BL(E), I_E, L(z,z) = L_{z^2}) \subset [0, +\infty)$, for all $z \in E_{sym}$. In particular $\phi(z^2) \geq 0$ for all $z \in E_{sym}$. Thus, it follows that

$$\|x^2 + y^2\| \geq \phi(x^2 + y^2) \geq \phi(x^2).$$

Finally, taking supreme on $\{\phi \in S_{E^*} : \phi(u) = 1\}$, we conclude by (2) that

$$\|x^2 + y^2\| \geq \|x^2\|,$$

which shows that E_{sym} is a JB-algebra. □

Remark 2.4. *Let E be a J*B-triple. Suppose that the complexification of E admits a complex JB*-triple structure with a norm extending the original norm of E. Then E is clearly a real JB*-triple. Let us denote by $\widehat{E} := E \oplus iE$ the complexification of E, by τ the canonical conjugation on $\widehat{E}$ satisfying $E = \widehat{E}^\tau$ and let $x \in E$. Since $\widehat{E}$ is a complex JB*-triple, it follows that $L(x,x)$ is an hermitian operator with non-negative spectrum, and hence, by* [2, Lemma 38.3],

$$V(BL(\widehat{E}), I_{\widehat{E}}, L(x,x)) \subset [0,+\infty).$$

By Remark 2.1 we also have

$$V(BL(E), I_E, L(x,x)) = V(BL(\widehat{E}), I_{\widehat{E}}, L(x,x)) \subset [0,+\infty).$$

Thus it seems natural to add the axiom

$$V(BL(E), I_E, L(x,x)) \subset [0,+\infty) \text{ for all } x \in E \tag{3}$$

*to the structure of the J*B-triple E, in order to show that E is also a real JB*-triple.*

We can now state our main result which shows that a J*B-triple E with a unitary element is a real JB*-triple if and only if E satisfies axiom (3). Previously, we recall (c.f. [1]) that a J*B-algebra is a real Jordan algebra A with unit and an involution * equipped with a complete algebra norm such that $\|U_x(x^*)\| = \|x\|^3$ and $\|x^*x\| \leq \|x^*x + y^*y\|$ for all $x, y \in A$.

Theorem 2.5. *Let E be a J*B-triple with a unitary element u. The following assertions are equivalent*

- (a) $V(BL(E), I_E, L(x,x)) \subset [0,+\infty)$ *for all $x \in E$;*
- (b) *E is a J*B-algebra with product $x \circ y := \{x,u,y\}$ and involution $x^* := \{u,x,u\}$;*
- (c) *E is a real JB*-triple;*
- (d) *The complexification of E is a complex JB*-triple in some norm extending the original norm on E.*

Proof. $(a) \Rightarrow (b)$ As we have seen before, E is a real Jordan Banach algebra with product $x \circ y := \{x,u,y\}$ and involution $x^* := \{u,x,u\}$. We have also seen that $\|U_x(x^*)\| = \|x\|^3$, for all $x \in E$. Hence only the inequality

$$\|x^* \circ x\| \leq \|x^* \circ x + y^* \circ y\|$$

has to be shown for all $x, y \in E$.

By Proposition 2.2, it follows that

$$\|2z \circ z^*\| = v(E, u, 2z \circ z^*) = v(BL(E), I_E, L(z,z) + L(z^*, z^*)) \tag{4}$$

for all $z \in E$.

Let $\phi \in S_{E^*}$ with $\phi(u) = 1$. By hypothesis, $V(BL(E), I_E, L(x,x))$ is contained in $[0, +\infty)$ for all $x \in E$, then we can conclude by the identity $L(x,x) + L(x^*, x^*) = L_{2x \circ x^*}$ that $\phi(x \circ x*) \geq 0$ for all $x \in E$. Therefore, given $x, y \in E$ we have

$$\|x \circ x^* + y \circ y^*\| \geq \phi(x \circ x^* + y \circ y^*) \geq \phi(x \circ x^*).$$

Finally, taking supreme on $\{\phi \in S_{E^*} : \phi(u) = 1\}$, we conclude by (4) that

$$\|x^* \circ x\| \leq \|x^* \circ x + y^* \circ y\|.$$

$(b) \Rightarrow (c)$ By [1, Theorem 4.4] it follows that the norm on E can be extended to the complexification $\widehat{E} = E \oplus iE$ of E such that $(\widehat{E}, \circ, ^*, \|.\|)$ is a JB*-algebra. Actually [1, Theorem 4.4] shows that E is a real form of its complexification. Since every JB*-algebra is a complex JB*-triple ([9, Proposition 20.35]), we conclude that E is a real JB*-triple.

The implications $(c) \Rightarrow (d)$ and $(d) \Rightarrow (a)$ follow from [4, Proposition 2.2] and Remark 2.4, respectively.

□

The next definition is motivated by the above theorem.

Definition 2.6. *An* axiomatic JB*-triple *is a J*B-triple E satisfying the following fifth axiom*

(J*B 5) $V(BL(E), I_E, L(x,x)) \subset [0, +\infty)$ *for all* $x \in E$.

Clearly every real JB*-triple is an axiomatic JB*-triple and every axiomatic JB*-triple is a J*B-triple. By Theorem 2.5 we know that if E is a Banach Jordan triple with a unitary element then problem $(P1)$ has an affirmative solution for E if and only if E is a real JB*-triple if and only if E is an axiomatic JB*-triple.

It seems natural to ask if we have found an axiomatic definition of real JB*-triples, but for the moment, we do not know if every axiomatic JB*-triple is a real JB*-triple.

Now we deal with problem $(P2)$. By [4, Lemma 4.2 and Theorem 4.4] we know that the bidual of a real JB*-triple is a real JB*-triple with a separate weak*-continuous triple product. Thus the following corollary is derived from Theorem 2.5.

Corollary 2.7. *Let E be a J*B-triple with a unitary element. Suppose that $V(BL(E), I_E, L(x,x)) \subset [0, +\infty)$ for all $x \in E$, i.e., E is an axiomatic JB*-triple with a unitary element. Then the bidual of E is again a J*B-triple with a unitary element and separately weak*-continuous triple product.*

If E is a real JB*-triple which is also a Banach dual space, then E has a unique predual and the triple product of E is separately weak*-continuous (c.f. [7]). Now applying Theorem 2.5, we obtain the following corollary.

Corollary 2.8. *Let E be an axiomatic JB*-triple with a unitary element. Suppose that E is a Banach dual space. Then E has a unique predual and the triple product of E is separately weak*-continuous.*

We leave open the following problems

($P3$) Is the complexification of an axiomatic JB*-triple a complex JB*-triple in some norm extending the original norm? This problem is equivalent to ask whether an axiomatic JB*-triple is a real JB*-triple.

($P4$) Is the bidual of an axiomatic JB*-triple an axiomatic JB*-triple with a separately weak*-continuous triple product?

($P5$) Let E be an axiomatic JB*-triple which is also a Banach dual space. Does E have a unique predual? Is the triple product on E separately weak*-continuous?

We have shown that problems ($P3$), ($P4$), and ($P5$) have an affirmative solution whenever E has a unitary element.

References

[1] Alvermann, K.: Real normed Jordan algebras with involution, Arch. Math. **47**, 135-150 (1986).

[2] Bonsall, F. F. and Duncan, J.: *Complete Normed Algebras*, Springer-Verlag, New York 1973.

[3] Dang, T. and Russo, B.: Real Banach Jordan triples, Proc. Amer. Math. Soc. **122**, 135-145 (1994).

[4] Isidro, J. M., Kaup, W., and Rodríguez, A.: On real forms of JB*-triples, *Manuscripta Math.* **86**, 311-335 (1995).

[5] Kaup, W.: Algebraic characterization of symmetric complex Banach manifolds, *Math. Ann.* **228**, 39-64 (1977).

[6] Kaup, W.: A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces, *Math. Z.* **183**, 503-529 (1983).

[7] Martínez, J. and Peralta A.M.: Separate weak*-continuity of the triple product in dual real JB*-triples, *Math. Z.*, **234**, 635-646 (2000).

[8] Peralta, A. M. and Rodríguez, A.: Grothendieck's inequalities revisited, In Recent Progress in Functional Analysis: Proceedings of the International Functional Analysis Meeting on the Occasion of the 70th Birthday of Professor Manuel Valdivia, Valencia, Spain, July 3-7, 2000, (ed. by K.D. Bierstedt, J. Bonet, M. Maestre, J. Schmets), to appear. North-Holland 2001.

[9] Upmeier, H.: Symmetric Banach Manifolds and Jordan C*-algebras, Mathematics Studies 104, (Notas de Matemática, ed. by L. Nachbin) North Holland, 1985.

Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain.
E-mail address: aperalta@goliat.ugr.es