CONTINUITY OF DERIVATIONS OF BANACH-LIE ALGEBRAS

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ABSTRACT. In his monograph of 1972, P. De La Harpe proved the continuity of the derivations of the classical Banach-Lie algebras of compact operators on Hilbert spaces. In this note we provide an elementary proof of this result by using uniquely the algebraic fact that these Banach-Lie algebras are strongly prime and contain extremal elements.

As pointed out in [**3**], all Banach-Lie algebras of compact operators on Hilbert spaces [**1**] enjoy the following algebraic properties: they are strongly prime and contain extremal elements.

A Lie algebra L (over a field $\mathbb{F}$ of characteristic 0) is said to be *strongly prime* if it is prime, as a nonassociative algebra, and *non-degenerate:* $\operatorname{ad}_x^2 L = 0$ implies $x = 0$. A non-zero element $e \in L$ is an *extremal element* of L if $\operatorname{ad}_e^2 L = \mathbb{F}e$. It is a purely algebraic fact that the existence of an extremal element in a non-degenerate Lie algebra L yields a non-zero *socle* $\operatorname{Soc}(L)$, which is a minimal ideal if L is strongly prime [**2**, Theorem 2.5].

Return to the case of a Banach-Lie algebra L. If D is a derivation of L, set $\operatorname{Sep}(D)$ to denote the separating subspace of D. It is well-known that $\operatorname{Sep}(D)$ is a closed ideal of L and that D is continuous if, and only if, $\operatorname{Sep}(D) = 0$.

LEMMA. *The separating ideal* $\operatorname{Sep}(D)$ *of a derivation* D *of a non-degenerate Banach-Lie algebra* L *cannot contain extremal elements.*

Proof. Otherwise, by [**2**, Proposition 1.18] together with [**3**, Lemma 7.1], $\operatorname{Sep}(D)$ would contain a pair of extremal elements (e, f) producing an $\mathfrak{sl}_2$-triple, i.e., $[[e, f], e] = 2e$ and $[[e, f], f] = -2f$. Since $f \in \operatorname{Sep}(D)$, there exists a sequence $\{x_n\} \subset L$ such that $x_n \to 0$ and $Dx_n \to f$. Now $\operatorname{ad}_e^2 L = \mathbb{C}e$ implies that for each positive integer

n there is a complex number α_n such that $\mathrm{ad}_e^2 x_n = \alpha_n e$. By the continuity of the Lie product, $\alpha_n \to 0$. Since D is a derivation,

$$D[e,[e,x_n]] = [De,[e,x_n]] + [e,[De,x_n]] + [e,[e,Dx_n]],$$

and since $x_n \to 0$, the continuity of the Lie product yields

$$\lim \alpha_n De = \lim D(\alpha_n e) = \lim D(\mathrm{ad}_e^2 x_n) = \lim \mathrm{ad}_e^2 Dx_n = \mathrm{ad}_e^2 f$$

Hence $-2e = \mathrm{ad}_e^2 f = \lim \alpha_n De = 0$, which is a contradiction.

REMARK. An argument similar to that of the lemma was used in [**4**, Theorem 3.6] as a part of the proof of the continuity of the two components of any derivation on a primitive Banach-Jordan pair.

THEOREM. *Derivations of strongly prime Banach-Lie algebras with extremal elements, in particular, derivations of classical Banach-Lie algebras of compact operators on Hilbert spaces, are automatically continuous.*

Proof. Let D be a derivation of a strongly prime Banach-Lie algebra L with extremal elements. If $\mathrm{Sep}(D)$ were nonzero, then we would have by strong primeness of L and structure of the socle [**2**, Proposition 2.5(ii)] that $\mathrm{Soc}(L) \subset \mathrm{Sep}(D)$. Hence $\mathrm{Sep}(D)$ would contain an extremal element, which is a contradiction by the lemma. Therefore $\mathrm{Sep}(D) = 0$ and D is continuous.

References

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[**3**] A. Fernández López, Banach-Lie algebras with extremal elements, Quart. J. Math. **00** (2009), 1-15; doi:10.1093/qmath/hap032.

[**4**] A. Fernández López, H. Marhnine and C. Zarhouti, Derivations on Banach-Jordan pairs, Quart. J. Math. **52** (2001), 269-283.

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