A Taste of Jordan Algebras

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Dedicated to the memory of Jake and Florie –
in mathematico parentis

To Jake (the Doktor-Vater) for his mathematical influence on my research, and to Florie (the Doktor-Mutter) for helping me (and all Jake's students) to get to know him as a warm human being. Future histories of mathematics should take into account the role of Doktor-Mutters in the fostering of mathematics.

1. Introduction

On several occasions I and colleagues have found ourselves teaching a 1-semester course for students at the second year of graduate study in mathematics who want to gain a general perspective on Jordan algebras, their structure and their role in mathematics, or want to gain direct experience with nonassociative algebra. These students typically have a solid grounding in first-year graduate algebra and the Artin-Wedderburn theory of associative algebras, and a few have been introduced to Lie algebras (perhaps even Cayley algebras, in an offhand way), but otherwise they have not seen any nonassociative algebras. Most of them will not go on to do research in nonassociative algebra, so the course is not primarily meant to be a training or breeding ground for research, though the instructor often hopes one or two will be motivated to pursue the subject further.

This text is meant to serve as an accompaniment to such a course. It is designed first and foremost to be read. It is a direct mathematical conversation between the author and a reader whose mind (as far as nonassociative algebra goes) is a tabula rasa. In keeping with the tone of a private conversation, I give more heuristic material than is common in books at this level (pep talks, philosophical pronouncements on the proper way to think about certain concepts, random historical anecdotes, offhand mention of some mathematicians who have contributed to our understanding of Jordan algebras, etc.), and employ a few English words which do not standardly appear in mathematical works. It is important for the reader to develop a visceral intuitive feeling for the subject, to view the mathematics as a living and active thing: to see isomorphisms as cloning maps, isotopes as subtle rearrangements of an algebra's DNA, radicals as pathogens to be isolated and removed by radical surgery, annihilators as biological agents for killing off elements, Peircers as mathematical enzymes ("Jordan-ase") which break an algebra down into its Peirce spaces. Like Charlie Brown's kite-eating trees, Jordan theory has Zel'manov's tetrad-eating ideals (though we shall stay clear of these carnivores in our book).

The book is intended for students to read on their own without assistance by a teacher. In particular, I have tried to make the proofs complete and understandable, giving much more heuristic and explanatory comment than is usual in graduate texts. To help the reader through the proofs in Parts III, IV (and the proof-sketches in Part II, Chapter 8), I have tried to give each important result or formula a mnemonic label, so that when I refer to an earlier result, instead of saying "by Formula 21.3(i), which of course you will remember, ..." I can say “by Nuclear Slipping 21.3(i)”, hoping to trigger long-repressed memories of a formula involving nuclear elements of alternative algebras.

While I wind up doing most of the talking, there is some room in Parts III and IV for the reader to participate (and stay mathematically fit) by doing exercises. The Exercises give slight extensions, or alternate proofs, of results in the text, and are placed immediately after the results; they give practice in proving variations on the previous mathematical theme . At the end of each chapter I gather a few problems and questions. The Problems usually take the form “Prove that something-or-other”; they involve deeper investigations or lengthier digressions than exercises, and develop more extensive proof skills on a new theme. The Questions are more open-ended, taking the form “What can you say about something-or-other” without giving a hint which way the answer goes; they develop proof skills in uncharted territories, in composing a mathematical theme from scratch (most valuable for budding researchers). Hints are given at the back of the book for some of the exercises, problems, and questions (though these should be consulted only after a good-faith effort to prove them).

Part I is in the nature of an extended colloquium talk, a brief survey of the life and times of Jordan algebras, to provide appreciation of the role Jordan algebras play on the broader stage of mathematics. I indicate several applications to other areas of mathematics: Lie algebras, differential geometry, and projective geometry. Since the students at this level cannot be assumed to be familiar with all these areas, the description has to be a bit loose; readers can glean from this partjust enough respect and appreciation to sanction and legitimate their investment in reading further.

Part II is designed to provide an overview of Jordan structure theory in its historical context. It gives a general historical survey from the origins in quantum mechanics in 1934 to Efim Zel’manov’s breathtaking description of arbitrary simple algebras in 1983 (which later played a role in his Fields Medal work on the Burnside Problem). I give precise definitions and examples, but omit proofs. In keeping with its nature, I have not included any exercises.

Parts III and IV are designed to provide direct experience with nonassociativity, and either one (in conjunction with Part I) could serve as a basis for a one-semester course. Throughout, I stick to linear Jordan algebras over rings of scalars containing 1/2, but give major emphasis to the quadratic point of view.

Part III gives a development of Jacobson’s classical structure theory for Jordan algebras with capacity, in complete detail and with full proofs. It is suitable for a one-semester course aiming to introduce students to the methods and techniques of nonassociative algebra. The details of Peirce decompositions, Peirce relations, and coordinatization theorems are the key tools leading to the Classical Structure Theorem.

Part IV gives a full treatment of Zel'manov's Exceptional Theorem, that the only simple i-exceptional Jordan algebras are the Albert algebras, closing the historical search for an exceptional setting for quantum mechanics. This part is much more concerned with understanding and translating to the Jordan setting some classical ideas of associative theory, including primitivity; it is suitable for a one-semester course aiming to introduce students to the modern methods of Jordan algebras. The ultrafilter argument, that if primitive systems come in only a finite number of flavors then a prime system must come in one of those pure flavors, is covered in full detail; ultrafilters provide a useful tool that many students at this level are unacquainted with.

I have dedicated the book to Nathan and Florie Jacobson, both of whom passed away during this book's long gestation period. They had an enormous influence on my mathematical development. I am greatly indebted to my colleague Kurt Meyberg, who carefully read through Part III and made many suggestions which vastly improved the exposition. I am also deeply indebeted to my colleague Wilhelm Kaup, who patiently corrected many of my misconceptions about the role of Jordan theory in diferential geometry, improving the exposition in Part I and removing flagrant errors. My colleague John Faulkner helped improve my discussion of applications to projective geometries. I would also like to thank generations of graduate students at Virginia who read and commented upon the text, especially my students Jim Bowling, Bernard Fulgham, Dan King, and Matt Neal.

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General Typographical Conventions

• Rings of scalars (unital, commutative, associative rings) are indicated by capital Greek letters Φ, Ω. Scalars are denoted by lower case Greek letters: $\alpha, \beta, \gamma, \ldots$ Almost all our algebraic systems will be algebras or modules over a fixed ring of scalars Φ, which will almost always contain an element $\frac{1}{2}$.

• Mere sets are indicated by italic capital letters X, Y, Z at the end of the alphabet, index sets also by I, J, S.

• Modules and linear spaces are denoted by italic capital letters: $A, B, C, J, V, W, \ldots$. The zero subspace will be denoted by boldface $\mathbf{0}$ to distinguish it from the element (operator, vector, or scalar) 0. This signals a subtle and not-too-important distinction between the *set* $\mathbf{0} = \{0\}$ consisting of a single element zero, and the *element* itself. The *range* $f(\mathrm{A})$ of some function on a set A will always be a set, while the *value* $f(a)$ will be an element.

• Algebraic systems are denoted by letters in small caps: general linear algebras by $\mathrm{A}, \mathrm{B}, \mathrm{C}$, ideals by $\mathrm{I}, \mathrm{J}, \mathrm{K}$. Associative algebras are indicated by D when they appear as coordinates for Jordan algebras. Jordan algebras are indicated by $\mathrm{J}, \mathrm{J}_i, \mathrm{J}'$, etc.

• Maps or functions between sets or spaces are denoted by italic lower case letters $f, g, h, \ldots$, morphisms between algebraic systems often by lower case Greek letters $\varphi, \sigma, \tau, \rho$, sometimes upper case italic letters T, S.

• Functors and functorial constructions are denoted by script capital letters $\mathcal{F}, \mathcal{G}, \mathcal{H}, \mathcal{T}, \ldots$.

Specific Notations

• The identity map on a set X is denoted by $\mathbf{1}_X$. The projection of a set on a quotient set is denoted by $\pi : X \to X/\sim$; the coset or equivalence class of $x \in X$ is denoted by $\pi(x) = \overline{x} = [x]$.

• Cartesian products of sets are denoted $X \times Y$. Module direct sums are indicated by $V \oplus W$. *Algebra direct sums, where multiplication as well as addition is performed componentwise, are written* $\mathrm{A} \boxplus \mathrm{B}$ *to distinguish them from mere module direct sums.*

• Subsets are denoted $X \subseteq Y$, with strict inclusion denoted by $X \subset Y$ or $X < Y$; subspaces of linear spaces are denoted $A \leq B$, while kernels (submodules of modules, ideals of algebras) are indicated by $\mathrm{B} \triangleleft \mathrm{A}$. Left, right, and $*$-ideals of algebras are denoted by $\mathrm{I} \triangleleft_\ell \mathrm{A}$, $\mathrm{I} \triangleleft_r \mathrm{A}$, $\mathrm{B} \overset{*}{\triangleleft} \mathrm{A}$.

• Involutions on algebras are indicated by a star $*$ (though involutions on coordinate algebras of matrix algebras are often denoted $x \to \bar{x}$). $\mathcal{H}(\mathrm{A}, *)$ denotes the hermitian elements $x^* = x$ of an algebra A under an involution $*$.

• Products in algebras are denoted by $x \cdot y$ or just xy (especially for associative products); the special symbol $x \bullet y$ is used for the bilinear product in Jordan algebras. The left and right multiplication operators by an element x in a linear algebra are denoted L_x, R_x or ℓ_x, r_x or λ_x, ρ_x. The quadratic and trilinear products in Jordan algebras are denoted by $U_x y$ and $\{x, y, z\}$, with operators $U_x, V_{x,y}$ (in Jordan triples $P_x, L_{x,y}$, in Jordan pairs $Q_x, D_{x,y}$).

• Unit elements of algebras are denoted by 1. We will speak of a *unit element* and *unital algebras* rather than identity element and algebras with identity; we will reserve the term *identity* for *identical relations* or *laws* (such as the Jordan identity or associative law). $\widehat{A}$ will denote the formal unital hull Φ-algebra $\Phi 1 \oplus A$ obtained by formal adjunction of a unit element.

• $n \times n$ matrices and hermitian matrices are denoted $\mathcal{M}_n$ and $\mathcal{H}_n$. Matrices are denoted by $X = (x_{ij})$; their traces and determinants are denoted by $tr(X)$, $det(X)$ respectively, and the transpose is indicated by X^{tr}. If the matrix entries come from a ring with involution, the *adjoint* (the conjugate transpose with ij-entry $\overline{x_{ji}}$) is denoted rather anonymously by a star, X^*.

• Blackboard bold is used for the standard systems $\mathbb{N}$ (natural numbers $1, 2, \ldots$), $\mathbb{I}$ (the even-more-natural numbers $0, 1, 2, \ldots$ used as indices or cardinals), $\mathbb{Z}$ (the ring of integers), the fields $\mathbb{Q}$ (rational numbers), $\mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers), the real division rings $\mathbb{H}$ (Hamilton's quaternions), $\mathbb{K}$ (Cayley's octonions), $\mathbb{O}$ (split octonions), $\mathbb{A}$ (the Albert algebra, a formally-real exceptional Jordan algebra).

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