ASSOCIATIVE GEOMETRIES. I: GROUDS, LINEAR RELATIONS AND GRASSMANNIANS

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Abstract. We define and investigate a geometric object, called an *associative geometry*, corresponding to an associative algebra (and, more generally, to an associative pair). Associative geometries combine aspects of *Lie groups* and of *generalized projective geometries*, where the former correspond to the Lie product of an associative algebra and the latter to its Jordan product. A further development of the theory encompassing *involutive* associative algebras will be given in subsequent work [BeKi09].

Introduction

What is the geometric object corresponding to an *associative* algebra? The question may come as a bit of a surprise: the philosophy of Noncommutative Geometry teaches us that, as soon as an algebra becomes noncommutative, we should stop looking for associated point-spaces, such as manifolds or varieties. Nevertheless, we raise this question, but aim at something different than Noncommutative Geometry: we do not try to generalize the relation between, say, commutative associative algebras and algebraic varieties, but rather look for an analog of the one between *Lie algebras* and *Lie groups*. Namely, every associative algebra $\mathbb{A}$ gives rise to a Lie algebra $\mathbb{A}^-$ with commutator bracket $[x,y] = xy - yx$, and thus can be seen as a "Lie algebra with some additional structure". Since the geometric object corresponding to a Lie algebra should be a Lie group (the unit group $\mathbb{A}^\times$, in this case), the object corresponding to the *associative* algebra, called an "associative geometry", should be some kind of "Lie group with additional structure". To get an idea of what this additional structure might be, consider the decomposition

$$xy = \frac{xy + yx}{2} + \frac{xy - yx}{2} =: x \bullet y + \frac{1}{2}[x,y]$$

of the associative product into its symmetric and skew-symmetric parts. The symmetric part is a *Jordan* algebra, and the additional structure will be related to the geometric object corresponding to the Jordan part. As shown in [Be02], the "geometric Jordan object" is a *generalized projective geometry*. Therefore, we expect an associative geometry to be some sort of mixture of projective geometry and Lie groups. Another hint is given by the notion of *homotopy* in associative algebras:

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an associative product xy really gives rise to a family of associative products

$$x \cdot_a y := xay$$

for any fixed element a, called the *a-homotope*. Therefore we should rather expect to deal with a whole family of Lie groups, instead of looking just at *one* group corresponding to the choice $a = 1$.

0.1. **Grassmannian grouds.** The following example gives a good idea of the kind of geometries we have in mind. Let W be a vector space or module over a commutative field or ring $\mathbb{K}$, and for a subspace $E \subset W$, let C^E denote the set of all subspaces of W complementary to E. It is known that C^E is, in a natural way, an affine space over $\mathbb{K}$. We prove that a similar statement is true for arbitrary intersections $C^E \cap C^F$ (Theorem 1.2): they are either empty, or they carry a natural "affine" group structure. By this we mean that, after fixing an arbitrary element $Y \in C^E \cap C^F$, there is a natural (in general noncommutative) group structure on $C^E \cap C^F$ with unit element Y. The construction of the group law is very simple: for $X, Z \in C^E \cap C^F$, we let $X \cdot Z := (P^E_X - P^Z_F)(Y)$, where, for any complementary pair (U, V), P^U_V is the projector onto V with kernel U. Since $X \cdot Z$ indeed depends on X, E, Y, F, Z, we write it also in pentary form

$$\Gamma(X, E, Y, F, Z) := (P^E_X - P^Z_F)(Y). \tag{0.1}$$

The reader is invited to prove the group axioms by direct calculations. The proofs are elementary, however, the associativity of the product, for example, is not obvious at a first glance.

Some special cases, however, are relatively clear. If $E = F$, and if we then identify a subspace U with the projection P^E_U, then it is straightforward to show that the expression $\Gamma(X, E, Y, E, Z)$ in C^E is equivalent to the expression $P^E_X - P^E_Y + P^E_Z$ in the space of projectors with kernel E, and we recover the classical affine space structure on C^E (see Theorem 1.2). On the other hand, if E and F happen themselves to be complementary, then any common complement of E and F may be identified with the graph of a bijective linear map $E \to F$, and hence $C^E \cap C^F$ is identified with the set $\mathrm{Iso}(E, F)$ of linear isomorphisms between E and F. Fixing an origin Y in this set fixes an identification of E and F, and thus identifies $C^E \cap C^F$ with the general linear group $\mathrm{Gl}_{\mathbb{K}}(E)$.

Summing up, the collection of groups $C^E \cap C^F$, where (E, F) runs through $\mathrm{Gras}(W) \times \mathrm{Gras}(W)$, the direct product of the Grassmannian of W with itself, can be seen as some kind of interpolation, or deformation between general linear groups and vector groups, encoded in Γ. The pentary map Γ has remarkable properties that will lead us to the axiomatic definition of associative geometries.

0.2. **Grouds and semigrouds.** To eliminate the dependence of the group structures $C^E \cap C^F$ on the choice of unit element Y, we now recall the "affine" or "base point free" concept of a group. There are several equivalent versions, going under different names such as *torsor*, *heap*, *flock*, *herd*, *principal homogeneous space*, *abstract coset*, *pregroup* or others. We follow B. Schein ([Sch62] and personal communications) and use the term *groud*. The idea is quite simple (see Appendix A for details): if, for a given group G with unit element e, we want to "forget the unit element", we consider G with the ternary product

$$G \times G \times G \to G; \quad (x, y, z) \mapsto (xyz) := xy^{-1}z.$$

As is easily checked, this map has the following properties: for all $x, y, z, u, v \in G$,

(G1) $$(xy(zuv)) = ((xyz)uv)\,,$$

(G2) $$(xxy) = y = (yxx)\,.$$

Conversely, given a set G with a ternary composition having these properties, for any element $x \in G$ we get a group law on G with unit x by letting $a \cdot_x b := (axb)$ (the inverse of a is then (xax)) and such that $(abc) = ab^{-1}c$ in this group. (This observation is stated explicitly by Certaine in [Cer43], based on earlier work Prüfer, Baer, and others.) Thus the affine concept of the group G is a set G with a ternary map satisfying (G1) and (G2); this is precisely the structure we call a *groud.*

One advantage of the groud concept, compared to other, equivalent notions mentioned above, is that it admits two natural and important extensions. On the one hand, a direct check shows that in any groud the relation

(G3) $$(xy(zuv)) = (x(uzy)v) = ((xyz)uv)\,,$$

called the *para-associative law*, holds (note the reversal of arguments in the middle term). Just as groups are generalized by semigroups, grouds are generalized by *semigrouds* which are simply sets with a ternary map satisfying (G3). By work dating back at least to that of V.V. Vagner, *e.g.* [Va66], it is already known that this concept has important applications in geometry and algebra.

On the other hand, *restriction to the diagonal* in a groud gives rise to an interesting product $m(x, y) := (xyx)$. The map $\sigma_x : y \mapsto m(x, y)$ is just inversion in the group (G, x). If G is a Lie groud (defined in the obvious way), then (G, m) is a *symmetric space* in the sense of Loos [Lo69].

0.3. **Grassmannian semigrouds.** One of the remarkable properties of the pentary map Γ defined above is that it admits an "algebraic continuation" from the subset $D(\Gamma) \subset \mathcal{X}^5$ of 5-tuples from the Grassmannian $\mathcal{X} = \mathrm{Gras}(W)$ where it was initially defined to *all* of $\mathcal{X}^5$. The definition given above requires that the pairs (E, X) and (F, Z) are complementary. On the other hand, fixing an arbitrary complementary pair (E, F), there is another natural ternary product: with respect to the decomposition $W = E \oplus F$, subspaces $X, Y, Z, \ldots$ of W can be considered as *linear relations between E and F*, and can be composed as such: $ZY^{-1}X$ is again a linear relation between E and F. Since $ZY^{-1}X$ depends on E and on F, we get another map

$$\Gamma(X, E, Y, F, Z) := XY^{-1}Z.$$

Looking more closely at the definition of this map, one realizes that there is a natural extension of its domain for *all* pairs (E, F), and that on $D(\Gamma)$ this new definition of Γ coincides with the earlier one given by (0.1) (Theorem 2.3). Moreover, for any fixed pair (E, F), the ternary product

$$(XYZ) := \Gamma(X, E, Y, F, Z)$$

turns the Grassmannian $\mathcal{X}$ into a semigroud. The list of remarkable properties of Γ does not end here – we also have *symmetry properties* with respect to the Klein 4-group acting on the variables (X, E, F, Z), certain interesting *diagonal values* relating the map Γ to *lattice theoretic properties* of the Grassmannian (Theorem 2.4) as well as *self-distributivity* of the product, reflecting the fact that all partial maps of Γ are *structural*, i.e., compatible with the whole structure (Theorem 2.7). Together, these properties can be used to give an axiomatic definition of an associative geometry (Chapter 3).

0.4. **Correspondence with associative algebras and pairs.** Taking the Lie functor for Lie groups as model, we wish to define a multilinear *tangent object* attached to an associative geometry at a given base point. A *base point* in $\mathcal{X}$ is a fixed complementary (we say also *transversal*) pair (o^+, o^-). The pair of abelian groups $(\mathbb{A}^+, \mathbb{A}^-) := (C^{o^-}, C^{o^+})$ then plays the rôle of a pair of "tangent spaces", and the rôle of the Lie bracket is taken by the following pair of maps:

$$f^\pm : \mathbb{A}^\pm \times \mathbb{A}^\mp \times \mathbb{A}^\pm \to \mathbb{A}^\pm; \quad (x, y, z) \mapsto \Gamma(x, o^+, y, o^-, z).$$

One proves that $f^\pm$ are trilinear (Theorem 3.4). Since the maps $f^\pm$ come from a semigroud, they form an *associative pair*, i.e., they satisfy the para-associative law (see Appendix B). Conversely, one can construct, for every associative pair, an associative geometry having the given pair as tangent object (Theorem 3.5). The prototype of an associative pair are operator spaces, $(\mathbb{A}^+, \mathbb{A}^-) = (\mathrm{Hom}(E, F), \mathrm{Hom}(F, E))$, with trilinear products $f^+(X, Y, Z) = XYZ$, $f^-(X, Y, Z) = ZYX$. They correspond precisely to Grassmannian geometries $\mathcal{X} = \mathrm{Gras}(E \oplus F)$ with base point $(o^+, o^-) = (E, F)$.

Associative unital algebras are associative pairs of the form $(\mathbb{A}, \mathbb{A})$; in the example just mentioned, this corresponds to the special case $E = F$. In this example, the unit element e of $\mathbb{A}$ corresponds to the diagonal $\Delta \subset E \oplus E$, and the subspaces (E, Δ, F) are mutually complementary. On the geometric level, this translates to the existence of a *transversal triple* (o^+, e, o^-). Thus the correspondence between associative geometries and associative pairs contains as a special case the one between associative geometries with transversal triples and unital associative algebras (Theorem 3.7).

0.5. **Further topics.** Since associative algebras play an important rôle in modern mathematics, the present work is related to a great variety of topics and leads to many new problems located at the interface of geometry and algebra. We mention some of them in the final chapter of this work, without attempting to be exhaustive. In particular, in Part II of this work ([BeKi09]) we will extend the theory to *involutive* associative algebras (Topic (2) mentioned in Chapter 4).

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Notation. Throughout this work, $\mathbb{K}$ denotes a commutative unital ring and $\mathbb{B}$ an associative unital $\mathbb{K}$-algebra, and we will consider *right* $\mathbb{B}$-modules $V, W, \ldots$. We think of $\mathbb{B}$ as "base ring", and the letter $\mathbb{A}$ will be reserved for other associative

$\mathbb{K}$-algebras such as $\mathrm{End}_{\mathbb{B}}(W)$. For a first reading, one may assume that $\mathbb{B} = \mathbb{K}$; only in Theorem 3.7 the possibility to work over non-commutative base rings becomes crucial.

When viewing submodules as elements of a Grassmannian, we will frequently use lower case letters to denote them, since this matches our later notation for abstract associative geometries. However, we will also sometimes switch back to the upper case notation we have already used whenever it adds clarity.

1. Grassmannian grouds

The *Grassmannian* of a right $\mathbb{B}$-module W is the set $\mathcal{X} = \mathrm{Gras}(W) = \mathrm{Gras}_{\mathbb{B}}(W)$ of all $\mathbb{B}$-submodules of W. If $x \in \mathcal{X}$ and $a \in \mathcal{X}$ are complementary ($W = x \oplus a$), we will write $x \top a$ and call the pair (x, a) *transversal*. We write $C_a := a^\top := \{x \in \mathcal{X} \mid x \top a\}$ for the set of all complements of a and

$$C_{ab} := a^\top \cap b^\top$$

for the set of common complements of $a, b \in \mathcal{X}$. We think of $a^\top$ and C_{ab} (which may or may not be empty) as "open chart domains" in $\mathcal{X}$. The following discussion makes this more precise.

1.1. Connected components and base points.

Connectedness. We define an equivalence relation in $\mathcal{X}$: $x \sim y$ if there is a finite sequence of "charts joining x and y", i.e.: $\exists a_0, a_1, \ldots, a_k$ such that $a_0 = x$, $a_k = y$ and

$$\forall i = 0, \ldots, k-1 : \quad C_{a_i, a_{i+1}} \neq \emptyset.$$

The equivalence classes of this relation are called *connected components of* $\mathcal{X}$. We say that $x \in \mathcal{X}$ is *isolated* if its connected component is a singleton. If $\mathbb{B} = \mathbb{K}$ and $\mathbb{K}$ is a field, then connected components are never reduced to a point (unless $x = 0$ or $x = W$). For instance, the connected components of $\mathrm{Gras}(\mathbb{K}^n)$ are the Grassmannians $\mathrm{Gras}_p(\mathbb{K}^n)$ of subspaces of a fixed dimension p (indeed, two subspaces of the same dimension p in $\mathbb{K}^n$ always admit a common complement, hence sequences of length 1 always suffice in the above condition: one then says that the geometry is *stable*).

Base points and pair geometries. A *base pair* or *base point* in $\mathcal{X}$ is a fixed transversal pair, often denoted by (o^+, o^-). If (o^+, o^-) is a base point, then in general o^+ and o^- belong to different connected components, which we denote by $\mathcal{X}^+$ and $\mathcal{X}^-$. For instance, in the Grassmann geometry $\mathrm{Gras}(\mathbb{K}^n)$ over a field $\mathbb{K}$, if o^+ is of dimension p, then o^- has to be of dimension $q = n - p$, and hence they belong to different components unless $p = q = \frac{n}{2}$.

More generally, we may consider certain subgeometries of $\mathcal{X}$, namely pairs $(\mathcal{X}^+, \mathcal{X}^-)$ such that $\mathcal{X}^\pm \subset \mathcal{X}$ are subsets such that, for every $x \in \mathcal{X}^\pm$, the set $x^\top$ is non-empty and belongs to $\mathcal{X}^\mp$. We refer to $(\mathcal{X}^+, \mathcal{X}^-)$ as a *pair geometry*.

For instance, if $W = \mathbb{B}$, then $\mathcal{X}$ is the *space of right ideals in* $\mathbb{B}$. Fix an idempotent $e \in \mathbb{B}$ and let $o^+ := e\mathbb{B}$, $o^- = (1-e)\mathbb{B}$ and $\mathcal{X}^\pm$ the set of all right ideals in $\mathbb{B}$ that are isomorphic to $o^\pm$ and have a complement isomorphic to $o^\mp$. Then $(\mathcal{X}^+, \mathcal{X}^-)$ is a pair geometry.

Transversal triples and spaces of the first kind. We say that $\mathcal{X}$ is *of the first kind* if there exists a triple (a, b, c) of mutual transversal elements, and *of the second kind* else. Clearly, a, b, c then all belong to the same connected component of $\mathcal{X}$; taking (a, c) as base point (o^+, o^-), we thus have $\mathcal{X}^+ = \mathcal{X}^-$. Note that $W = a \oplus c$ with $a \cong b \cong c$, so W is "of even dimension". For instance, the Grassmann geometry $\mathrm{Gras}(\mathbb{K}^n)$ over a field $\mathbb{K}$ is of the first kind if and only if n is even, and the preceding example of a pair geometry of right ideals is of the first kind if and only if o^+ and o^- are isomorphic as $\mathbb{B}$-modules. In other words, $\mathbb{B}$ is a direct sum of two copies of some other algebra, and $\mathcal{X}^+ = \mathcal{X}^-$ is the *projective line over this algebra*, *cf.* [BeNe05].

1.2. **Basic operators and the product map Γ.** If x and a are two complementary $\mathbb{B}$-submodules, let $P^a_x : W \to W$ be the projector onto x with kernel a. Since a and x are $\mathbb{B}$-modules, this map is $\mathbb{B}$-linear. The relations

$$P^a_x \circ P^a_z = P^a_x, \qquad P^a_x \circ P^b_x = P^b_x, \qquad P^z_b \circ P^a_z = 0$$

will be constantly used in the sequel. For a $\mathbb{B}$-linear map $f : W \to W$, we denote by $[f] := f \mod \mathbb{K}^\times$ be its projective class with respect to invertible scalars from $\mathbb{K}$. By 1 we denote the (class of) the identity operator on W. We define the following operators : if $a \top x$ and $z \top b$, define the *middle multiplication operator* (motivation for this terminology will be given below)

$$M_{xabz} := [P^a_x - P^z_b],$$

and if $a \top x$ and $y \top b$, define the *left multiplication operator*

$$L_{xayb} := [1 - P^x_a P^b_y]$$

and if $a \top y$ and $z \top b$, define the *right multiplication operator*

$$R_{aybz} := L_{zbya} = [1 - P^z_b P^a_y].$$

For a scalar $s \in \mathbb{K}$ and a transversal pair (x, a), the *dilation operator* is defined by

$$\delta^{(s)}_{xa} := [sP^x_a + P^a_x] = [1 - (1 - s)P^x_a] = [s1 + (1 - s)P^a_x].$$

Note that the dilation operator for the scalar -1 is also a middle multiplication operator:

$$\delta^{(-1)}_{xa} = [-P^x_a + P^a_x] = M_{xaax},$$

and it is induced by a reflection with respect to a subspace. Also, $\delta^{(1)}_{xa} = 1$ and $\delta^{(0)}_{xa} = [P^a_x]$.

Proposition 1.1.

i) (Symmetry) M_{xabz} is invariant under permutations of indices by the Klein 4-group:

$$M_{xabz} = M_{axzb} = M_{bzxa} = M_{zbax}.$$

ii) (Fundamental Relation) Whenever $u, x \top a$ and $v, z \top b$,

$$R_{aubz} L_{xavb} = M_{xabz} M_{uabv} = L_{xavb} R_{aubz}.$$

iii) *(Diagonal values) If $x \in C_{ab}$,*

$$L_{xaxb} = 1 = R_{axbx},$$

and, for all $u \in C_{ab}$ and $z \top b$,

$$M_{uabz}(u) = z = R_{aubz}(u).$$

iv) *(Compatibility) If $x \top a$, $y \top b$, $z \top b$ and C_{ab} is not empty, then*

$$L_{xayb}(z) = M_{xabz}(y),$$

and if $x \top a$, $z \top b$, $y \top a$ and C_{ab} is not empty, then

$$M_{xabz}(y) = R_{aybz}(x).$$

v) *(Invertibility) Let $(x, a, y, b, z) \in \mathcal{X}^5$ such that $x, y, z \in C_{ab}$. Then the operators*

$$L_{xayb}, \quad M_{xabz}, \quad R_{aybz}$$

are invertible, with inverse operators, respectively,

$$L_{yaxb}, \quad M_{zabx}, \quad R_{azby}.$$

Proof. (i):

$$M_{xabz} = [P_x^a - P_b^z] = [P_b^z - P_x^a] = M_{bzxa}.$$

Since this is the only place where we really use that $[f] = [-f]$, for simplicity of notation, we henceforth omit the brackets [ ].

$$M_{zbax} = P_z^b - P_a^x = (1 - P_b^z) - (1 - P_x^a) = M_{xabz}$$

(ii): Using in the second line that $P_a^x P_v^b P_b^z P_u^a = 0$:

$$\begin{aligned} L_{xavb} R_{aubz} &= (1 - P_a^x P_v^b)(1 - P_b^z P_u^a) \\ &= 1 - P_a^x P_v^b - P_b^z P_u^a \\ &= 1 - (1 - P_x^a)(1 - P_b^v) - P_b^z P_u^a \\ &= P_x^a + P_b^v - P_x^a P_b^v - P_b^z P_u^a \\ &= (P_x^a - P_b^z)(P_u^a - P_b^v) \\ &= M_{xabz} M_{uabv}. \end{aligned}$$

The relation $R_{aubz} L_{xavb} = M_{xabz} M_{uabv}$ now follows from (i).

(iii): $L_{xaxb} = 1 = R_{axbx}$ is clear. Fix an element $u \in C_{ab}$. Then, for all $z \top b$,

$$M_{uabz}(u) = M_{zbau}(u) = (P_z^b - P_a^u)(u) = P_z^b(u) = z$$

since both u and z are complements of b. Similarly,

$$R_{aubz}(u) = (1 - P_b^z P_u^a)(u) = (1 - P_b^z)(u) = P_z^b(u) = z.$$

(iv): By (ii), $M_{xabz} M_{uaby} = L_{xayb} R_{aubz}$. Apply this operator to $u \in C_{ab}$ and use that, by (3), $M_{uaby}(u) = y$ and $R_{aubz}(u) = z$. One gets

$$M_{xabz}(y) = M_{xabz} M_{uaby}(u) = L_{xayb} R_{aubz}(u) = L_{xayb}(z).$$

Via the symmetry relation (i), the second equality can also be written $M_{zbax}(y) = L_{zbya}(x)$ and hence is equivalent to the first one.

(v): Since $L_{xaxb} = 1 = R_{axbx}$, the fundamental relation (ii) implies $M_{xabz} M_{zaby} = L_{xayb}$ and

$$M_{xabz} M_{zabx} = L_{xaxb} = 1,$$

hence M_{xabz} is invertible with inverse M_{zabx}. The other relations are proved similarly. □

Remark. We will prove in Chapter 2 by different methods that the assumption $C_{ab} \neq \emptyset$ in (iv) is unnecessary.

Definition (of the product map Γ). *We define a map $\Gamma : D(\Gamma) \to \mathcal{X}$ on the following domain of definition: let*

$$\begin{aligned} D_L &:= \{(x,a,y,b,z) \in \mathcal{X}^5 \mid x \top a \ \textit{ and } \ y \top b\} \\ D_R &:= \{(x,a,y,b,z) \in \mathcal{X}^5 \mid y \top a \ \textit{ and } \ z \top b\} \\ D_M &:= \{(x,a,y,b,z) \in \mathcal{X}^5 \mid x \top a, \ z \top b \ \textit{ and } \ C_{ab} \neq \emptyset\} \\ D(\Gamma) &:= D_L \cup D_R \cup D_M \, , \end{aligned}$$

and define $\Gamma : D(\Gamma) \to \mathcal{X}$ *by*

$$\Gamma(x,a,y,b,z) := \begin{cases} L_{xayb}(z) & \textit{if} \quad (x,a,y,b,z) \in D_L \\ R_{aybz}(x) & \textit{if} \quad (x,a,y,b,z) \in D_R \\ M_{axbz}(y) & \textit{if} \quad (x,a,y,b,z) \in D_M \, . \end{cases}$$

This is well-defined: if $(x,a,y,b,z) \in D_L \cap D_R$, then $y \in C_{ab}$, hence C_{ab} is not empty and the preceding proposition implies that

$$L_{xayb}(z) = M_{xabz}(y) = R_{aybz}(x).$$

Similar remarks apply to the cases $(x,a,y,b,z) \in D_L \cap D_M$ or $(x,a,y,b,z) \in D_R \cap D_M$. The pentary map Γ explains our terminology and notation: L_{xayb} is the left multiplication operator, acting on the last argument z, and similarly R and M denote right and middle multiplication operators. From the definition it follows easily that the symmetry relation

$$\Gamma(x,a,y,b,z) = \Gamma(z,b,y,a,x)$$

holds for all $(x,a,y,b,z) \in D(\Gamma)$. On the other hand, the relation

$$\Gamma(x,a,y,b,z) = \Gamma(a,x,y,z,b)$$

holds if $(x,a,y,z,b) \in D_M$; but at present it is somewhat complicated to show that this relation is valid on all of $D(\Gamma)$ (this will follow from the results of Chapter 2). As to the “diagonal values”, for $x \in C_{ab}$ we have

$$\Gamma(x,a,x,b,z) = z = \Gamma(z,b,x,a,x) \, .$$

If we assume just $a \top x$ and $b \top z$, then we can only say in general that

$$\Gamma(x,a,x,b,z) = (1 - P_b^z P_x^a)(x) = P_z^b(x) \subset z \, .$$

If $a, b \top x$ and $b \top z$, then, thanks to the symmetry relation $M_{xabz} = M_{axzb}$,

$$M_{xabz}(a) = \Gamma(x,a,a,b,z) = \Gamma(a,x,a,z,b) = b \, . \tag{1.1}$$

Definition (of the dilation map Π_s). *Fix* $s \in \mathbb{K}$. *Let*

$$D(\Pi_s) := \{(x,a,z) \in \mathcal{X}^3 \mid x \top a \ \textit{ or } \ z \top a\}$$

and define a ternary map $\Pi_s : D(\Pi_s) \to \mathcal{X}$ *by*

$$\Pi_s(x,a,z) := \begin{cases} \delta_{xa}^{(s)}(z) & \text{if} \quad x \top a \\ \delta_{za}^{(1-s)}(x) & \text{if} \quad z \top a. \end{cases}$$

As above, this map is well-defined. The symmetry relation

$$\Pi_s(x,a,y) = \Pi_{1-s}(y,a,x)$$

follows easily from the definition. Note that, if s is invertible in $\mathbb{K}$ and $x \top a$, then the dilation operator $\delta_{xa}^{(s)}$ is invertible with inverse $\delta_{xa}^{(s^{-1})}$.

1.3. **Grassmannian grouds and their actions.** Recall from §0.2 and Appendix A the definition and elementary properties of *grouds.*

Theorem 1.2. *The Grassmannian geometry* $(\mathcal{X}; \Gamma, \Pi_r)$ *defined in the preceding subsection has the following properties:*

i) For $a, b \in \mathcal{X}$ *fixed,* C_{ab} *with product*

$$(xyz) := \Gamma(x,a,y,b,z)$$

is a groud (which will be denoted by U_{ab}*). In particular, for a triple* (a,y,b) *with* $y \in C_{ab}$*,* C_{ab} *is a group with unit* y *and multiplication* $xz = \Gamma(x,a,y,b,z)$*.*

ii) The map Γ *is symmetric under the permutation* $(15)(24)$ *(reversal of arguments):*

$$\Gamma(x,a,y,b,z) = \Gamma(z,b,y,a,x)$$

In other words, U_{ab} *is the opposite groud of* U_{ba} *(same set with reversed product). In particular, the groud* $U_a := U_{aa}$ *is commutative.*

iii) The commutative groud U_a *is the underlying additive groud of an affine space: for any* $a \in \mathcal{X}$*,* U_a *is an affine space over* $\mathbb{K}$*, with additive structure given by*

$$x +_y z = \Gamma(x,a,y,a,z),$$

(sum of x *and* z *with respect to the origin* y*), and action of scalars given by*

$$sy + (1-s)x = \Pi_s(x,y)$$

(multiplication of y *by* s *with respect to the origin* x*).*

Proof. (i) Let us show first that C_{ab} is stable under the ternary map (xyz). Let $x, y, z \in C_{ab}$ and consider the bijective linear map $g := M_{xabz}$. We show that $g(y) \in C_{ab}$. By equation (1.1), we have the "diagonal values" $M_{xabz}(a) = b$ and $M_{xabz}(b) = a$. Thus, if y is complementary to a and b, $g(y)$ is complementary both to $g(a) = b$ and to $g(b) = a$, which means that $g(y) \in C_{ab}$.

The associativity follows immediately from the "fundamental relation" (Proposition 1.1(ii)):

$$(xv(yuz)) = L_{xavb} R_{aubz}(y) = R_{aubz} L_{xavb}(y) = ((xvy)uz),$$

and the idempotent laws from

$$(xxy) = L_{xaxb}(y) = 1(y) = y, \quad (yxx) = R_{axbx}(y) = 1(y) = y.$$

Thus C_{ab} is a groud.

(ii) This has already been shown in the preceding section.

(iii) The set C_a is the space of complements of a. It is well-known that this is an affine space over $\mathbb{K}$. Let us recall how this affine structure is defined (see, *e.g.*, [Be04]): elements $v \in C_a$ are in one-to-one correspondence with projectors of the form P^a_v. Then, for $u, v, w \in U_a$, the structure map $(u, v, w) \mapsto u +_v w$ in the affine space C_a is given by associating to (u, v, w) the point corresponding to the projector $P^a_u - P^a_v + P^a_w$, and the structure map $(v, w) \mapsto r \cdot_v w = (1 - r)v + rw$ by associating to (v, w) the point corresponding to the projector $rP^a_u + (1 - r)P^a_v$. Now write $y = P^a_y(W)$; then we have

$$\begin{aligned}\Gamma(x, a, y, a, z) &= (P^a_x - P^z_a)(y) \\ &= (P^a_x - 1 + P^a_z)P^a_y(W) \\ &= (P^a_x - P^a_y + P^a_z)(W),\end{aligned}$$

and

$$\Pi_s(x, a, y) = ((1 - s)P^a_x + s1)(z) = ((1 - s)P^a_x + sP^a_z)(W),$$

proving that (iii) describes the usual affine structure of C_a. □

Homomorphisms. We think of the maps $\Gamma : D(\Gamma) \to \mathcal{X}$ and $\Pi_r : D(\Pi_r) \to \mathcal{X}$ as pentary, resp. ternary "product maps" defined on (parts of) direct products $\mathcal{X}^5$, resp. $\mathcal{X}^3$. Thus we have basic categorical notions just as for groups, rings, modules etc.: *homomorphisms* are maps $g : \mathcal{X} \to \mathcal{Y}$ preserving transversality ($x \top y$ implies $g(x) \top g(y)$) and such that, for all 5-tuples in $D(\Gamma)$, resp. triples in $D(\Pi_r)$,

$$\begin{aligned}g\left(\Gamma(u, c, v, d, w)\right) &= \Gamma\left(g(u), g(c), g(v), g(d), g(w)\right) , \\ g\left(\Pi_r(u, c, v)\right) &= \Pi_r\left(g(u), g(c), g(v)\right) .\end{aligned}$$

Essentially, this means that all restrictions of g,

$$U_{ab} \to U_{g(a),g(b)}, \qquad U_a \to U_{g(a)},$$

are usual homomorphisms (of grouds, resp. of affine spaces). We may summarize this by saying that g is "locally linear" and "compatible with all local group structures".

Theorem 1.3. *Assume $x, y, z \in U_{ab}$. Then the operators*

$$M_{xabz} : \mathcal{X} \to \mathcal{X}, \quad L_{xayb} : \mathcal{X} \to \mathcal{X}, \quad R_{aybz} : \mathcal{X} \to \mathcal{X}$$

are automorphisms of the geometry $(\mathcal{X}, \Gamma, \Pi_r)$, and the groups (U_{ab}, y) act on $\mathcal{X}$ by automorphisms both from the left and from the right via

$$(U_{ab}, y) \times \mathcal{X} \to \mathcal{X}, \quad (x, z) \mapsto L_{xayb}(z) = \Gamma(x, a, y, b, z),$$

respectively

$$\mathcal{X} \times (U_{ab}, y) \to \mathcal{X}, \quad (x, z) \mapsto R_{aybz}(x) = \Gamma(x, a, y, b, z).$$

For fixed (a, y, b), the left and right actions commute.

Proof. The construction of the product map Γ is "natural" in the sense that all elements of $\mathrm{Gl}_{\mathbb{B}}(W)$ (acting from the left on W, commuting with the right $\mathbb{B}$-module structure) act by automorphisms of $(\mathcal{X}, \Gamma)$, just by ordinary push-forward of sets. This follows immediately from the relation $g \circ P^a_x = P^{g(a)}_{g(x)} \circ g$. In particular, the invertible linear operators M_{xabz}, L_{xayb} and R_{aybz} induce automorphisms of $(\mathcal{X}, \Gamma)$.

Now fix $y \in U_{ab}$ and consider it as the unit in the group (U_{ab}, y). The claim on the left action amounts to the identities $L_{yayb} = \mathrm{id}$ (which we already know) and, for all $x, x' \in U_{ab}$ and all $z \in \mathcal{X}$,

$$\Gamma(x, a, y, b, \Gamma(x', a, y, b, z)) = \Gamma(\Gamma(x, a, y, b, x'), a, y, b, z).$$

First, note that, if z is "sufficiently nice", i.e., such that the fundamental relation (Proposition 1.1(ii)) applies, then this holds indeed. We will show in Chapter 2 that the identity in question holds very generally, and this will prove our claim. Therefore we leave it as a (slightly lengthy) exercise to the interested reader to prove the claim in the present framework. The claims concerning the right action are proved in the same way, and the fact that both actions commute is precisely the content of the fundamental relation (Proposition 1.1(ii)) □

Inner automorphisms. We call automorphisms of the geometry defined by the preceding theorem *inner automorphisms*, and the group generated by them the *inner automorphism group*. Note that middle multiplications M_{xabz} are honest automorphisms of the geometry $(\mathcal{X}, \Gamma)$, although they are *anti*-automorphisms of the groud U_{ab}; this is due to the fact that they exchange a and b. On the other hand, L_{xayb} and R_{aybz} are automorphisms of the whole geometry *and* of U_{ab}.

Note also that the action of the groups U_{ab} is of course very far from being regular on its orbits, except on U_{ab} itself. For instance, a and b are fixed points of these actions, since $\Gamma(x, a, y, b, b) = b$ and $\Gamma(x, a, y, b, a) = a$.

Finally, the statements of the preceding two theorems amount to certain algebraic identities for the multiplication map Γ. This will be taken up in Chapter 2, where we will not have to worry about domains of definition.

1.4. **Affine picture of the groud U_{ab}.** It is useful to have "explicit formulas" for our map Γ. Such formulas can be obtained by introducing "coordinates" on $\mathcal{X}$ in the following way (see [Be04]). First of all, choose a base point (o^+, o^-) and consider the pair geometry $(\mathcal{X}^+, \mathcal{X}^-)$, where $\mathcal{X}^\pm$ is the space of all submodules isomorphic to $o^\pm$ and having a complement isomorphic to $o^\mp$. We identify $\mathcal{X}^+$ with injections $x : o^+ \to W$ of $\mathbb{B}$ right-modules, modulo equivalence under the action of the group $G := \mathrm{Gl}(o^+)$ ($x \cong x \circ g$, where g acts on o^+ on the left), and $\mathcal{X}^-$ with $\mathbb{B}$-linear surjections $a : W \to o^+$ (modulo equivalence $a \cong g \circ a$ for $g \in G$). Equivalence classes are denoted by $[x]$, resp. $[a]$.

Proposition 1.4. *The following explicit formulae hold.*

i) if $x \top a$ and $z \top b$ (middle multiplication), then

$$\Gamma\Big([x], [a], [y], [b], [z]\Big) = \Big[x(ax)^{-1}ay - y + z(bz)^{-1}by\Big], \tag{1.2}$$

ii) if $a \top x$ and $b \top y$ (left multiplication), then

$$\Gamma\Big([x], [a], [y], [b], [z]\Big) = \Big[x(ax)^{-1}ay(by)^{-1}(bz) - y(by)^{-1}(bz) + z\Big], \tag{1.3}$$

iii) if $a \top y$ and $b \top z$ (right multiplication), then

$$\Gamma\Big([x], [a], [y], [b], [z]\Big) = \Big[x - y(ay)^{-1}ax + z(bz)^{-1}za(ay)^{-1}ax\Big]. \tag{1.4}$$

Proof. The right hand side of (1.2) is a well-defined element of $\mathcal{X}$, as is seen by replacing x by $x \circ g$, resp. y by $y \circ g$, z by $z \circ g$ and a by $g \circ a$, b by $g \circ b$. Note that $[x]$ and $[a]$ are transversal if and only if $ax : o^+ \to o^+$ is invertible. Now, the operator

$$x(ax)^{-1}a : W \to o^+ \to W$$

has kernel a and image x and is idempotent, therefore it is P^a_x. Similarly, we see that $z(bz)^{-1}b$ is P^b_z, and hence the right hand side is induced by the operator $P^a_x - 1 + P^b_z = M_{xabz}$. Similarly, we see that the right hand side of (1.3) is induced by the linear operator

$$P^a_x P^b_y - P^b_y + 1 = (P^a_x - 1)P^b_y + 1 = 1 - P^x_a P^b_y = L_{axby}$$

and the one of (1.4) by $1 - P^a_y + P^b_z P^a_y = 1 + (P^b_z - 1)P^a_y = 1 - P^z_b P^a_y = R_{zbya}$. $\square$

As usual in projective geometry, the projective formulas from the preceding result may be affinely re-written: if $y \top b$, we may affinize by taking $([y],[b])$ as base point (o^+, o^-): we write $W = o^- \oplus o^+$; then injections $x : o^+ \to W$, $z : o^+ \to W$ that are transversal to the first factor can be identified with column vectors (by normalizing the second component to be the identity operator on o^+)

$$x = \begin{pmatrix} X \\ 1 \end{pmatrix}, \qquad z = \begin{pmatrix} Z \\ 1 \end{pmatrix}$$

(columns with $X, Z \in \mathrm{Hom}(o^+, o^-)$). In other terms, x and z are graphs of linear operators $X, Z : o^+ \to o^-$. Surjections $a : W \to o^+$ that are transversal to the second factor correspond to row vectors $(A, 1)$ (row with $A \in \mathrm{Hom}(o^-, o^+)$). Note, however, that the kernel of $(A, 1)$ is determined by the condition $Au + v = 0$, i.e., $v = -Au$, and hence a is the graph of $-A : o^- \to o^+$. Therefore we write $a = (-A, 1)$. The base point $y = o^+$ is the column $(0, 1)^t$, and the base point $b = o^-$ is the row $(0, 1)$. Since $ax = (-A, 1)(X, 1)^t = 1 - AX$, a and x are transversal iff $1 - AX : o^+ \to o^+$ is an invertible operator (in Jordan theoretic language: the pair (X, A) is *quasi-invertible*, cf. [Lo75]). Using this, any of the three formulas from the preceding proposition leads to the "affine picture":

$$\Gamma(x, a, y, b, z) = \left[\begin{pmatrix} X \\ 1 \end{pmatrix} (1 - AX)^{-1} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} Z \\ 1 \end{pmatrix} \right] = \left[\begin{pmatrix} -ZAX + X + Z \\ 1 \end{pmatrix} \right].$$

Finally, identifying x with X, y with Y and so on, we may write

$$\Gamma(X, A, O^+, O^-, Z) = X - ZAX + Z\,.$$

This formula is interesting in many respects: it is affine in all three variables, and the product ZAX from the associative pair $\big(\mathrm{Hom}(o^+, o^-), \mathrm{Hom}(o^-, o^+)\big)$ shows up. We will give conceptual explanations of these facts later on. Also, it is an easy exercise to check directly that $(X, Z) \mapsto X - ZAX + Z$ defines an associative product on $\mathrm{Hom}(o^+, o^-)$ and induces a group structure on the set of elements X such that $1 - AX$ is invertible.

Other "rational" formulas. More generally, having fixed (o^+, o^-), we may write a, b as row-, and x, y, z as column vectors, and then we get the general formula

$$\Gamma(X, A, Y, B, Z) =$$
$$\left[\begin{pmatrix} X \\ 1 \end{pmatrix} (1 - AX)^{-1}(1 - AY) - \begin{pmatrix} Y \\ 1 \end{pmatrix} + \begin{pmatrix} Z \\ 1 \end{pmatrix} (1 - BZ)^{-1}(1 - BY) \right],$$

which is (the class of) a vector with second component ("denominator")

$$D := (1 - AX)^{-1}(1 - AY) - 1 + (1 - BZ)^{-1}(1 - BY),$$

and first component ("numerator")

$$N := X(1 - AX)^{-1}(1 - AY) - Y + Z(1 - BZ)^{-1}(1 - BY),$$

so that the affine formula is $\Gamma(X, A, Y, B, Z) = ND^{-1}$. Besides the above choice ($Y = O^+$, $B = O^-$), another reasonable choice is just $B = O^-$, leading to

$$\Gamma(X, A, Y, O^-, Z) = X - (Y - Z)(1 - AY)^{-1}(1 - AX).$$

Similarly, for $Y = O^+$ we get formulas that, in case $A = B$, correspond to well-known Jordan theoretic formulas for the quasi-inverse. Such formulas show that, if we work in finite dimension over a field, Γ is a rational map in the sense of algebraic geometry, and if we work in a topological setting over topological fields or rings, then Γ will have smoothness properties similar to the ones described in [BeNe05].

Case of a geometry of the first kind. Assume there is a transversal triple, say, (o^+, e, o^-). We may assume that e is the diagonal in $W = o^- \oplus o^+$. Take, in the formulas given above, $a = 0 = (0, 1)$, $b = \infty = (1, 0)$, $y = (1, 1)^t$, $ax = (0, 1)(X, 1)^t = 1$, $bz = (1, 0)(Z, 1)^t = Z$, $ay = 1$, $by = 1$, so we get

$$\Gamma(X, 0, e, \infty, Z) = \left[\begin{pmatrix} X \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} Z \\ 1 \end{pmatrix} Z^{-1} \right] = \left[\begin{pmatrix} X \\ Z^{-1} \end{pmatrix} \right] = \left[\begin{pmatrix} XZ \\ 1 \end{pmatrix} \right],$$

and hence the affine picture is the algebra $\mathrm{End}_{\mathbb{B}}(o^+)$ with its usual product. Taking $a = \infty$, $b = 0$ gives the opposite of the usual product. Replacing e by $y = \{(v, Yv) \mid Y : o^+ \to o^-\}$ (graph of an invertible linear map Y), we get the affine picture

$$\Gamma(X, 0, Y, \infty, Z) = \left[\begin{pmatrix} XY^{-1}Z \\ 1 \end{pmatrix} \right].$$

1.5. **Affinization: the transversal case.** If a and b are arbitrary, then in general the groud U_{ab} will be empty. Therefore we look at the pair (U_a, U_b).

Theorem 1.5. *For all $a, b \in \mathcal{X}$, we have*

$$\Gamma(U_a, a, U_b, b, U_a) \subset U_a, \quad \Gamma(U_b, a, U_a, b, U_b) \subset U_b.$$

In other words, the maps

$$U_a \times U_b \times U_a \to U_a; \quad (x, y, z) \mapsto (xyz)^+ := L_{xayb}(z) = \Gamma(x, a, y, b, z),$$
$$U_b \times U_a \times U_b \to U_b; \quad (x, y, z) \mapsto (xyz)^- := R_{aybz}(x) = \Gamma(x, a, y, b, z)$$

are well-defined. If, moreover, $a \top b$*, then both maps are trilinear, and they form an* associative pair*, i.e., they satisfy the para-associative law (cf. Appendix B)*

$$(xy(uvw)^{\pm})^{\pm} = ((xyu)^{\pm}vw)^{\pm} = (x(vuy)^{\mp}w)^{\pm}.$$

Proof. Assume that $x \top a$ and $y \top b$. By a direct calculation, we will show that $L_{xayb}(U_a) \subset U_a$. Let us write L_{xayb} in matrix form with respect to the decomposition $W = a \oplus x$. The projectors P^x_a and P^b_y can be written

$$P^x_a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P^b_y = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

whith $\alpha \in \mathrm{End}(a)$, $\beta \in \mathrm{Hom}(x, a)$, etc. Thus

$$L_{xayb} = 1 - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1-\alpha & -\beta \\ 0 & 1 \end{pmatrix}.$$

Let $z \in U_a$; it can be written as the graph $\{(Zv, v) | \, v \in x\}$ of a linear operator $Z : x \to a$. Since

$$L_{xayb} \begin{pmatrix} Zv \\ v \end{pmatrix} = \begin{pmatrix} 1-\alpha & -\beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Zv \\ v \end{pmatrix} = \begin{pmatrix} (1-\alpha)Zv - \beta v \\ v \end{pmatrix},$$

$L_{xayb}(z)$ is the graph of the linear operator $(1-\alpha)Z - \beta : x \to a$, and hence is again transversal to a, so $(\quad)^+$ is well-defined. By symmetry, it follows that $(\quad)^-$ is well-defined. Moreover, the calculation shows that $z \mapsto (xyz)^+$ is affine (we will see later that this map is actually affine with respect to all three variables, see Corollary 2.9).

Now assume that $a \top b$, and write L_{xayb} in matrix form with respect to the decomposition $W = a \oplus b$. The projectors P^a_x and P^b_y can be written

$$P^a_x = \begin{pmatrix} 0 & X \\ 0 & 1 \end{pmatrix}, \quad P^b_y = \begin{pmatrix} 1 & 0 \\ Y & 0 \end{pmatrix}$$

where $X \in \mathrm{Hom}(b, a)$ and $Y \in \mathrm{Hom}(a, b)$. We get

$$L_{xayb} = 1 - \left(1 - \begin{pmatrix} 0 & X \\ 0 & 1 \end{pmatrix}\right) \begin{pmatrix} 1 & 0 \\ Y & 0 \end{pmatrix} = 1 - \begin{pmatrix} 1 - XY & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} XY & 0 \\ 0 & 1 \end{pmatrix}$$

and, writing $z \in U_a$ as a graph $\{(Zv, v) | v \in b\}$, we get

$$L_{xayb} \begin{pmatrix} Zv \\ v \end{pmatrix} = \begin{pmatrix} XY & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Zv \\ v \end{pmatrix} = \begin{pmatrix} XYZv \\ v \end{pmatrix},$$

hence $L_{xayb}(z)$ is the graph of $XYZ : b \to a$. Thus, with $V^+ = U_a \cong \mathrm{Hom}(b, a)$, $V^- = U_b \cong \mathrm{Hom}(a, b)$, the first ternary map is given by

$$V^+ \times V^- \times V^+ \to V^+, \quad (X, Y, Z) \mapsto XYZ.$$

Similarly, one shows that the second ternary map is given by

$$V^- \times V^+ \times V^- \to V^-, \quad (X, Y, Z) \mapsto ZYX.$$

This pair of maps is the prototype of an associative pair (see Appendix B). $\square$

At this stage, the appearance of the trilinear expression ZYX, resp. ZAX, both in the affine pictures of the map from the preceding theorem and in the preceding section, related by the identity

$$X - (X - ZAX + Z) + Z = ZAX, \tag{1.5}$$

looks like a pure coincidence. See Chapter 3 (Lemma 3.3) for a conceptual explanation.

2. Grassmannian semigrouds

In this chapter we extend the definition of the product map Γ onto all of $\mathcal{X}^5$, and we show that the most important algebraic identities extend also. We use notation and general notions explained in the first section of the preceding chapter.

2.1. **Composition of relations.** Recall that, if $A, B, C, \ldots$ are any sets, we can *compose relations*: for subsets $x \subset A \times B$, $y \subset B \times C$,

$$y \circ x := yx := \{(u,w) \in A \times C \mid \exists v \in B: \ (u,v) \in x, (v,w) \in y\}.$$

Composition is associative: both $(z \circ y) \circ x$ and $z \circ (y \circ x)$ are equal to

$$z \circ y \circ x = \{(u,w) \in A \times D \mid \exists (v_1, v_2) \in y: \ (u, v_1) \in x, (v_2, w) \in z\}. \tag{2.1}$$

If x and y are graphs of maps X, resp. Y ($v = Xu$, $w = Yv$) then $y \circ x$ is the graph of YX ($w = Yv = YXu$). The *reverse relation* of x is

$$x^{-1} := \{(w,v) \in B \times A \mid (v,w) \in x\}.$$

We have $(yx)^{-1} = x^{-1}y^{-1}$, and if x is the graph of a bijective map, then x^{-1} is the graph of its inverse map. For $x, y, z \subset A \times B$, we get another relation between A and B by $zy^{-1}x$. Obviously, this ternary composition satisfies the para-associative law, and hence relations between A and B form a semigroud. Letting $W := A \times B$, we have the explicit formula

$$\begin{aligned} zy^{-1}x &= \left\{ \omega = (\alpha', \beta') \in W \,\middle|\, \begin{array}{c} \exists \eta = (\alpha'', \beta'') \in y: \\ (\alpha', \beta'') \in x, (\alpha'', \beta') \in z \end{array} \right\} \\ &= \left\{ \omega \in W \,\middle|\, \begin{array}{c} \exists \alpha', \alpha'' \in A, \exists \beta', \beta'' \in B, \exists \eta \in y, \exists \xi \in x, \exists \zeta \in z: \\ \omega = (\alpha', \beta'), \eta = (\alpha'', \beta''), \xi = (\alpha', \beta''), \zeta = (\alpha'', \beta') \end{array} \right\} \end{aligned}$$

2.2. **Composition of linear relations.** Now assume that $A, B, C, \ldots$ are *linear* spaces over $\mathbb{B}$ (i.e., right modules) and that all relations are *linear relations* (i.e., submodules of $A \oplus B$, etc.). Then $zy^{-1}x$ is again a linear relation. Identifying A with the first and B with the second factor in $W := A \oplus B$, the description of $zy^{-1}x$ given above can be rewritten, by introducing the new variables $\alpha := \alpha' - \alpha''$, $\beta := \beta' - \beta''$,

$$\begin{aligned} zy^{-1}x &= \left\{ \omega \in W \,\middle|\, \begin{array}{c} \exists \alpha', \alpha'' \in a, \exists \beta', \beta'' \in b, \exists \eta \in y, \exists \xi \in x, \exists \zeta \in z: \\ \omega = \alpha' + \beta', \eta = \alpha'' + \beta'', \xi = \alpha' + \beta'', \zeta = \alpha'' + \beta' \end{array} \right\} \\ &= \left\{ \omega \in W \,\middle|\, \begin{array}{c} \exists \alpha', \alpha \in a, \exists \beta', \beta \in b, \exists \eta \in y, \exists \xi \in x, \exists \zeta \in z: \\ \omega = \alpha' + \beta', \eta = \omega - \alpha - \beta, \xi = \omega - \beta, \zeta = \omega - \alpha \end{array} \right\}. \end{aligned}$$

In order to stress that the product $xy^{-1}z$ depends also on A and B, we will henceforth use lowercase letters a and b and write $W = a \oplus b$.

Lemma 2.1. *Assume $W = a \oplus b$ and let $x, y, z \in \mathrm{Gras}_{\mathbb{B}}(W)$. Then*

$$zy^{-1}x = \left\{ \omega \in W \,\middle|\, \begin{array}{c} \exists \xi \in x, \exists \alpha \in a, \exists \eta \in y, \exists \beta \in b, \exists \zeta \in z : \\ \omega = \zeta + \alpha = \alpha + \eta + \beta = \xi + \beta \end{array} \right\} .$$

Proof. Since $W = a \oplus b$, the first condition ($\exists \alpha' \in a, \beta' \in b$: $\omega = \alpha' + \beta'$) in the preceding description is always satisfied and can hence be omitted in the description of $zy^{-1}x$. Replacing α by $-\alpha$ and β by $-\beta$, the claim follows. $\square$

2.3. **The extended product map.** Motivated by the considerations from the preceding section, we now *define* the product map $\Gamma : \mathcal{X}^5 \to \mathcal{X}$ for *all* 5-tuples of the Grassmannian $\mathcal{X} = \mathrm{Gras}_{\mathbb{B}}(W)$ by

$$\Gamma(x, a, y, b, z) := \left\{ \omega \in W \,\middle|\, \begin{array}{c} \exists \xi \in x, \exists \alpha \in a, \exists \eta \in y, \exists \beta \in b, \exists \zeta \in z : \\ \omega = \zeta + \alpha = \alpha + \eta + \beta = \xi + \beta \end{array} \right\} .$$

We will show, among other things, that this notation is in keeping with the one introduced in the preceding chapter. Firstly, however, we collect various equivalent formulas for Γ. The three conditions

$$\begin{array}{rcl} \omega & = & \eta + \alpha + \beta \\ \omega & = & \beta + \xi \\ \omega & = & \alpha + \zeta \end{array} \tag{2.2}$$

can be re-written in various ways. For instance, subtracting the last two equations from the first one we get the equivalent conditions

$$\omega = -\eta + \xi + \zeta, \quad \omega = \beta + \xi, \quad \omega = \alpha + \zeta \tag{2.3}$$

and hence, replacing η by $-\eta$, we get

$$\Gamma(x, a, y, b, z) = \left\{ \omega \in W \,\middle|\, \begin{array}{c} \exists \xi \in x, \exists \alpha \in a, \exists \eta \in y, \exists \beta \in b, \exists \zeta \in z : \\ \omega = \zeta + \alpha = \xi + \eta + \zeta = \xi + \beta \end{array} \right\} .$$

Next, letting $\alpha' = -\alpha$ and $\beta' = -\beta$, conditions (2.2) are equivalent to

$$\eta = \omega + \alpha' + \beta', \quad \zeta = \omega + \alpha', \quad \xi = \omega + \beta', \tag{2.4}$$

and hence we get

$$\begin{aligned} \Gamma(x, a, y, b, z) &= \left\{ \omega \in W \,\middle|\, \begin{array}{c} \exists \xi \in x, \exists \alpha \in a, \exists \eta \in y, \exists \beta \in b, \exists \zeta \in z : \\ \eta = \omega + \alpha + \beta, \zeta = \omega + \alpha, \xi = \omega + \beta \end{array} \right\} \\ &= \left\{ \omega \in W \,\middle|\, \exists \beta \in b, \exists \alpha \in a : \omega + \alpha \in z, \omega + \alpha + \beta \in y, \omega + \beta \in x \right\} . \end{aligned}$$

The following lemma now follows by straightforward changes of variables:

Lemma 2.2. *For all $x, a, y, b, z \in \mathcal{X}$,*

$$\begin{aligned}\Gamma(x,a,y,b,z) &= \Big\{\omega \in W \,\Big|\, \exists \xi \in x, \exists \zeta \in z : \zeta + \omega \in a, \zeta + \omega + \xi \in y, \omega + \xi \in b\Big\} \\ &= \Big\{\omega \in W \,\Big|\, \exists \xi \in x, \exists \alpha \in a : \omega - \alpha \in z, \xi - \alpha \in y, \omega - \xi \in b\Big\} \\ &= \Big\{\omega \in W \,\Big|\, \exists \beta \in b, \exists \zeta \in z : \zeta - \omega \in a, \zeta - \beta \in y, \omega - \beta \in x\Big\} \\ &= \Big\{\omega \in W \,\Big|\, \exists \eta \in y, \exists \beta \in b : \omega - \eta - \beta \in a, \beta + \eta \in z, \omega - \beta \in x\Big\} \\ &= \Big\{\omega \in W \,\Big|\, \exists \eta \in y, \exists \zeta \in z : \omega + \zeta \in a, \zeta + \eta \in b, \omega + \zeta + \eta \in x\Big\}\end{aligned}$$

We refer to the descriptions of the lemma as the "(x,z)-", "(x,a)-description", and so on. The (a,b)-description is particularly useful for the proof of the theorem below. One may note that the only pairs of variables that cannot be used for such a description are (a,z) and (x,b), and that the signs in the terms appearing in these descriptions can be chosen positive if the pair is "homogeneous" (a subpair of (x,y,z) or of (a,b)), whereas for "mixed" pairs we cannot get rid of signs.

Theorem 2.3. *The map $\Gamma : \mathcal{X}^5 \to \mathcal{X}$ extends the product map defined in the preceding chapter, and has the following properties:*

(1) It is symmetric under the Klein 4-group:

(a) $$\Gamma(x,a,y,b,z) = \Gamma(z,b,y,a,x)\,,$$

(b) $$\Gamma(x,a,y,b,z) = \Gamma(a,x,y,z,b)\,.$$

(2) For any pair $(a,b) \in \mathcal{X}^2$, the product $(xyz) := \Gamma(x,a,y,b,z)$ on $\mathcal{X}^3$ satisfies the properties of a semigroud, that is,

$$\Gamma\big(x,a,u,b,\Gamma(y,a,v,b,z)\big) = \Gamma\big(x,a,\Gamma(v,a,y,b,u),b,z\big) = \Gamma\big(\Gamma(x,a,u,b,y),a,v,b,z\big)\,.$$

We will write $\mathcal{X}_{ab}$ for $\mathcal{X}$ equipped with this semigroud structure. Then the semigroud $\mathcal{X}_{ba}$ is the opposite semigroud of $\mathcal{X}_{ab}$; in particular, $\mathcal{X}_{aa}$ is a commutative semigroud, for any a.

Proof. (1) The symmetry relation (a) is obvious from the definition of Γ. Exchanging x and a amounts in the (x,a)-description to exchanging simultaneously z and b, hence the symmetry relation (b) follows.

For (2), we use the (a,b)-description: on the one hand,

$\Gamma\big(x,a,u,b,\Gamma(y,a,v,b,z)\big) =$

$$\begin{aligned}&= \left\{\omega \in W \,\middle|\, \begin{array}{c}\exists \alpha \in a, \exists \beta \in b : \\ \omega + \alpha \in \Gamma(y,a,v,b,z), \omega + \alpha + \beta \in u, \omega + \beta \in x\end{array}\right\} \\ &= \left\{\omega \in W \,\middle|\, \begin{array}{c}\exists \alpha \in a, \exists \beta \in b, \exists \alpha' \in a, \exists \beta' \in b : \\ \omega + \alpha + \beta \in u, \omega + \beta \in x, \omega + \alpha + \alpha' \in z, \\ \omega + \alpha + \alpha' + \beta' \in v, \omega + \alpha + \beta' \in y\end{array}\right\}\end{aligned}$$

On the other hand,

$$\Gamma\big(x,a,\Gamma(u,b,y,a,v),b,z\big) =$$
$$= \left\{ \omega \in W \,\middle|\, \begin{matrix} \exists \alpha'' \in a, \exists \beta'' \in b : \\ \omega + \alpha'' \in z, \omega + \alpha'' + \beta'' \in \Gamma(u,b,y,a,v), \omega + \beta'' \in x \end{matrix} \right\}$$
$$= \left\{ \omega \in W \,\middle|\, \begin{matrix} \exists \alpha'' \in a, \exists \beta'' \in b, \exists \alpha''' \in a, \exists \beta''' \in b : \\ \omega + \alpha'' \in z, \omega + \beta'' \in x, \omega + \alpha'' + \beta'' + \alpha''' \in u, \\ \omega + \alpha'' + \beta'' + \beta''' \in v, \omega + \alpha'' + \beta'' + \alpha''' + \beta''' \in y \end{matrix} \right\}$$

Via the change of variables $\alpha'' = \alpha + \alpha'$, $\alpha''' = \alpha'$, $\beta'' = \beta$, $\beta''' = \beta$, we see that these two subspaces of W are the same. The remaining equality is equivalent to the one just proved via the symmetry relation (a).

Next, we show that the new map Γ coincides with the old one on $D(\Gamma)$. Let us assume that $(x,a,y,b,z) \in D_L$, so $x \top a$ and $y \top b$. We use the (y,b)-description and let $\zeta := \eta + \beta$, whence $\eta = P_y^b \zeta$ and $\beta = P_b^y \zeta$. We get

$$\begin{aligned} \Gamma(x,a,y,b,z) &= \Big\{ \omega \in W \,\Big|\, \exists \eta \in y, \exists \beta \in b : \omega - \eta - \beta \in a, \beta + \eta \in z, \omega - \beta \in x \Big\} \\ &= \Big\{ \omega \in W \,\Big|\, \exists \zeta \in z : \omega - P_b^y(\zeta) \in x, \omega - \zeta \in a \Big\} \\ &= \Big\{ \omega \in W \,\Big|\, \exists \zeta \in z : P_x^a(\omega - \zeta) = 0, P_x^a(\omega - P_b^y(\zeta)) = \omega - P_b^y(\zeta) \Big\} \\ &= \Big\{ \omega \in W \,\Big|\, \exists \zeta \in z : P_x^a \zeta = P_x^a \omega, \omega = P_b^y \zeta + P_x^a \omega - P_x^a P_b^y \zeta \Big\} \\ &= \Big\{ \omega \in W \,\Big|\, \exists \zeta \in z : \omega = (P_b^y + P_x^a - P_x^a P_b^y)\zeta \Big\} \end{aligned}$$

and a straightforward calculation shows that

$$P_b^y + P_x^a - P_x^a P_b^y = 1 - P_a^x P_y^b = L_{xayb}$$

so that $\Gamma(x,a,y,b,z) = L_{xayb}(z)$. This proves that the old and new definitions of Γ coincide on D_L, and hence also on D_R by the symmetry relation. Now we show that the new map Γ coincides with the old one on D_M: assume $a \top x$ and $b \top z$ and use the (x,z)-description; let $\eta := \zeta - \omega + \xi$ and observe that $P_x^a \eta = P_x^a \xi = \xi$ (since $\zeta - \omega \in a$), and similarly $P_z^b \eta = \zeta$, whence $\omega = \zeta - \eta + \xi = (P_z^b - 1 + P_x^a)\eta$, and thus

$$\begin{aligned} \Gamma(x,a,y,b,z) &= \Big\{ \omega \in W \,\Big|\, \exists \xi \in x, \exists \zeta \in z : \zeta - \omega \in a, \zeta - \omega + \xi \in y, \omega - \xi \in b \Big\} \\ &= \Big\{ \omega \in W \,\Big|\, \exists \eta \in y : \omega = (P_z^b - 1 + P_x^a)\eta \Big\} , \end{aligned}$$

that is, $\omega = -M_{xabz}\eta$, and hence $\Gamma(x,a,y,b,z) = M_{xabz}(y)$. □

2.4. **Diagonal values.** We call *diagonal values* the values taken by Γ on the subset of $\mathcal{X}^5$ where at least two of the five variables x,a,y,b,z take the same value. There are two different kinds of behavior on such diagonals: for the diagonal $a = b$ (or, equivalently, $x = z$), we still have a rich algebraic theory which is equivalent to the *Jordan part* of our associative products; this topic is left for subsequent work (cf. Chapter 4). The three remaining diagonals ($x = y$, resp. $a = z$, resp. $b = z$) have an entirely different behavior: the algebraic operation Γ restricts in these cases to *lattice theoretic operations*, that is, can be expressed by intersections and sums of subspaces. We will use the lattice theoretic notation $x \wedge y = x \cap y$ and $x \vee y = x + y$. It is remarkable that two important aspects of projective geometry (the lattice theoretic and the Jordan theoretic) arise as a sort of "contraction" of the full map Γ, or, put differently, that they have a common "deformation", given by Γ.

Theorem 2.4. *The map $\Gamma : \mathcal{X}^5 \to \mathcal{X}$ takes the following diagonal values:*
(1) values on the "diagonal $x = y$": for all $(x, a, b, z) \in \mathcal{X}^4$,

$$\Gamma(x, a, x, b, z) = (z \vee (x \wedge a)) \wedge (b \vee x).$$

In particular, we get the following "subdiagonal values": for all x, a, y, b, z,
(i) subdiagonal $x = y = z$: $\Gamma(x, a, x, b, x) = x$ (law $(xxx) = x$ in $\mathcal{X}_{ab}$),
(ii) subdiagonal $x = y = a$: $\Gamma(x, x, x, b, z) = (z \vee x) \wedge (b \vee x)$
(iii) subdiagonal $x = y = a$ and $b = z$: $\Gamma(x, x, x, z, z) = z \vee x$
(iv) subdiagonal $x = y = b$: $\Gamma(x, a, x, x, z) = (z \vee (x \wedge a)) \wedge x$.
(v) subdiagonal $x = y = b$ and $a = z$: $\Gamma(x, a, x, x, a) = a \wedge x$
(vi) subdiagonal $x = y$, $a = z$: $\Gamma(x, a, x, b, a) = a \wedge (b \vee x)$
(vii) subdiagonal $x = y$, $a = b$: $\Gamma(x, a, x, a, z) = (z \vee (x \wedge a)) \wedge (x \vee a)$
(viii) subdiagonal $x = y$, $z = b$: $\Gamma(x, a, x, z, z) = z \vee (x \wedge a)$
(2) diagonal $a = z$: for all $(x, a, y, b) \in \mathcal{X}^4$,

$$\Gamma(x, a, y, b, a) = a \wedge (b \vee (x \wedge (y \vee a)))$$

In particular, on the subdiagonal $x = z = b$, we have, for all $x, a, y \in \mathcal{X}$,

$$\Gamma(x, a, y, x, a) = a \wedge x.$$

(3) diagonal $b = z$: for all $(x, a, y, b) \in \mathcal{X}^4$,

$$\Gamma(x, a, y, b, b) = b \vee (a \wedge (x \vee (y \wedge b)))$$

In particular, on the subdiagonal $b = z$, $x = a$, we have, for all $a, y, b \in \mathcal{X}$,

$$\Gamma(a, a, y, b, b) = b \vee a,$$

and on $a = b = z$: for all $x, a, y \in \mathcal{X}$, $\Gamma(x, a, y, a, a) = a$.

Proof. In the following proof, in order to avoid unnecessary repetitions, it is always understood that $\alpha \in a$, $\xi \in x$, $\beta \in b$, $\eta \in y$, $\zeta \in z$. In all three items, the determination of the "subdiagonal values" is a straightforward consequence, using the absorption laws $u \vee (u \wedge v) = u$, $u \wedge (u \vee v) = u$.

Now we prove (1) (diagonal $x = y$). Let $\omega \in \Gamma(x, a, x, b, z)$, then $\omega = \xi + \beta$, hence $\omega \in (x \vee b)$, and $\omega = \eta + \xi + \zeta$ with $v := \omega - \zeta = \eta + \xi \in x$ (since $x = y$). On the other hand, $v = \omega - \zeta = \alpha \in a$, whence $\omega = v + \zeta$ with $v \in (x \wedge a)$, proving one inclusion.

Conversely, let $\omega \in (z \vee (x \wedge a)) \wedge (b \vee x)$. Then $\omega = \beta + \xi = \alpha + \zeta$ with $\alpha \in (x \wedge a)$. Let $\eta := \xi - \alpha$. Then $\eta \in x$, and $\omega = \xi + \beta = \eta + \alpha + \beta$, hence $\omega \in \Gamma(x, a, x, b, z)$.

Next we prove (2) (diagonal $a = z$). Let $\omega \in \Gamma(x, a, y, b, a)$, then $\omega = \zeta + \alpha$ with $\zeta, \alpha \in z = a$, whence $\omega \in a$. Moreover, $\omega = \xi + \beta = \eta + \alpha + \beta$, with $\eta + \alpha = \xi \in x$ and $\eta + \alpha \in y \vee a$, whence $\omega \in b \vee (x \wedge (y \vee a))$.

Conversely, let $\omega \in a \wedge (b \vee (x \wedge (y \vee a)))$. Then $\omega \in b \vee (x \wedge (y \vee a))$, that is, $\omega = \beta + (\eta + \alpha)$ with $\xi := \eta + \alpha \in x$. Letting $\zeta := \omega - \alpha \in a$ (here we use that $\omega \in a$), we have $\omega = \zeta + \alpha$, proving that $\omega \in \Gamma(x, a, y, b, a)$.

The proof for (3) (diagonal $z = b$) is "dual" to the preceding one and will be left to the reader. $\square$

Remark. By arguments of the same kind as above, one can show that the diagonal value for $x = y$ (Part (1)) admits also another, kind of "dual", expression:

$$\Gamma(x, a, x, b, z) = (z \wedge (x \vee b)) \vee (a \wedge x). \tag{2.5}$$

The equality of these two expressions is equivalent to the *modular law*

$$\Gamma(x, a, x, x, z) = (z \wedge x) \vee (a \wedge x) = ((z \wedge x) \vee a) \wedge x. \tag{2.6}$$

Corollary 2.5. *(1) If $b \vee x = W$ and $a \wedge x = 0$, then $\Gamma(x, a, x, b, z) = z$.*
(2) If $a \vee y = W$ and $b \vee x = W$, then $\Gamma(x, a, y, b, a) = a$.
(3) If $x \wedge a = 0$ and $y \wedge b = 0$, then $\Gamma(x, a, y, b, b) = b$.

Proof. Straightforward consequences of the theorem, again using the absorption laws. $\square$

2.5. **Structural transformations and self-distributivity.** *Homomorphisms* between sets with pentary product maps Γ, Γ' are defined in the usual way, and may serve to define the category of Grassmannian geometries with their product maps Γ. We call this the "usual" category. There is another and often more useful way to turn them into a category which we call "structural":

Definition. *Let W, W' be two right $\mathbb{B}$-modules and $(\mathcal{X}, \Gamma)$, $(\mathcal{X}', \Gamma')$ their Grassmannian geometries. A* structural *or* adjoint *pair of transformations between $\mathcal{X}$ and $\mathcal{X}'$ is a pair of maps $f : \mathcal{X} \to \mathcal{X}'$, $g : \mathcal{X}' \to \mathcal{X}$ such that, for all $x, a, y, b, z \in \mathcal{X}$, $x', a', y', b', z' \in \mathcal{X}'$,*

$$f\big(\Gamma(x, g(a'), y, g(b'), z)\big) = \Gamma'(f(x), a', f(y), b', f(z)),$$
$$g\big(\Gamma'(x', f(a), y', f(b), z')\big) = \Gamma(g(x'), a, g(y'), b, g(z'))\,.$$

In other words, for fixed a, b, resp. a', b', the restrictions

$$f : \mathcal{X}_{g(a'),g(b')} \to \mathcal{X}'_{a',b'}\,, \quad g : \mathcal{X}'_{f(a),f(b)} \to \mathcal{X}_{a,b}$$

are homomorphisms of semigrouds. We will sometimes write (f, f^t) for a structural pair (although g need not be uniquely determined by f).

It is easily checked that the composition of structural pairs gives again a structural pair, and Grassmannian geometries with structural pairs as morphisms form a category. Isomorphisms, and, in particular, the automorphism group of $(\mathcal{X}, \Gamma)$, are essentially the same in the usual and in the structural categories, but the endomorphism semigroups may be very different. Roughly speaking, Grassmannian geometries tend to be "simple objects" in the usual category (hence morphisms tend to be either trivial or injective), whereas they are far from being simple in the structural category, so there are many morphisms. One way of constructing such morphisms is via ordinary $\mathbb{B}$-linear maps $f : W \to W'$, which induce maps between the corresponding Grassmannians $\mathcal{X} = \mathrm{Gras}(W)$ and $\mathcal{X}' = \mathrm{Gras}(W')$:

$$f_* : \mathcal{X} \to \mathcal{X}';\ x \mapsto f(x), \qquad f^* : \mathcal{X}' \to \mathcal{X};\ y \mapsto f^{-1}(y).$$

Note that, in general, these maps do not restrict to maps between connected components (for instance, f_* and f^* do not restrict to *everywhere* defined maps between projective spaces $\mathbb{P}W$ and $\mathbb{P}W'$ if f is not injective). We will show that (f_*, f^*) is an adjoint pair, as a special case of the following result:

Theorem 2.6. *Given a linear relation* $\mathbf{r} \subset W \oplus W'$, *let*

$$\begin{aligned} \mathbf{r}_* : \mathcal{X} \to \mathcal{X}'; \quad & x \mapsto \mathbf{r}(x) := \{\omega' \in W' \mid \exists \xi \in x : (\xi, \omega') \in \mathbf{r}\}, \\ \mathbf{r}^* : \mathcal{X}' \to \mathcal{X}; \quad & y \mapsto \mathbf{r}^{-1}(y) := \{\omega \in W \mid \exists \eta \in y : (\omega, \eta) \in \mathbf{r}\}. \end{aligned}$$

Then $(\mathbf{r}_*, \mathbf{r}^*)$ *is a structural pair of transformations between* $\mathcal{X}$ *and* $\mathcal{X}'$.

Proof. Using the (a, b)-description, on the one hand,

$$\begin{aligned} &\mathbf{r}_*\Gamma(x, \mathbf{r}^*a', y, \mathbf{r}^*b', z) = \\ &= \left\{ \omega' \in W' \,\middle|\, \exists \omega \in \Gamma(x, \mathbf{r}^*a', y, \mathbf{r}^*b', z) : (\omega, \omega') \in \mathbf{r} \right\} \\ &= \left\{ \omega' \in W' \,\middle|\, \begin{array}{c} \exists \omega \in W, \exists \alpha \in \mathbf{r}^*a', \exists \beta \in \mathbf{r}^*b' : \\ (\omega, \omega') \in \mathbf{r}, \omega + \alpha \in z, \omega + \beta \in x, \omega + \alpha + \beta \in y \end{array} \right\} \\ &= \left\{ \omega' \in W' \,\middle|\, \begin{array}{c} \exists \omega \in W, \exists \alpha' \in a', \exists \alpha \in W, \exists \beta' \in b', \exists \beta \in W : \\ (\omega, \omega') \in \mathbf{r}, (\alpha, \alpha') \in \mathbf{r}, (\beta, \beta') \in \mathbf{r}, \\ \omega + \alpha \in z, \omega + \beta \in x, \omega + \alpha + \beta \in y \end{array} \right\}, \end{aligned}$$

and on the other hand,

$$\begin{aligned} &\Gamma(\mathbf{r}_*x, a', \mathbf{r}_*y, b', \mathbf{r}_*z) = \\ &= \left\{ \omega' \in W' \,\middle|\, \begin{array}{c} \exists \alpha'' \in a', \exists \beta'' \in b' : \\ \omega' + \alpha'' \in \mathbf{r}^*z, \omega' + \beta'' \in \mathbf{r}^*x, \omega' + \alpha'' + \beta'' \in \mathbf{r}^*y \end{array} \right\} \\ &= \left\{ \omega' \in W' \,\middle|\, \begin{array}{c} \exists \alpha'' \in a', \exists \beta'' \in b', \exists \zeta \in z, \exists \xi \in x, \exists \eta \in y : \\ (\zeta, \omega' + \alpha'') \in \mathbf{r}, (\xi, \omega' + \beta'') \in \mathbf{r}, (\eta, \omega' + \alpha'' + \beta'') \in \mathbf{r} \end{array} \right\}. \end{aligned}$$

The subspaces of W determined by these two conditions are the same, as is seen by the change of variables

$$\zeta = \omega + \alpha, \ \xi = \omega + \beta, \ \eta = \omega + \alpha + \beta, \ \alpha'' = \alpha', \ \beta'' = \beta'$$

in one direction, and

$$\omega = \eta - \zeta - \xi, \alpha'' = \alpha', \ \beta'' = \beta', \ \alpha = \zeta - \omega = \eta - \xi, \ \beta = \xi - \omega = \eta - \zeta$$

in the other, and using that $\mathbf{r}$ is a linear subspace. □

Remark. The proof shows that the same result would hold if we had formulated the structurality property with respect to another "admissible" pair of variables instead of (a, b), for instance (y, b) or (x, z), by using the corresponding description. However, we prefer to distinguish the pair formed by the second and fourth variable in order to have the interpretation of structural transformations in terms of groud homomorphisms, for fixed (a, b).

Remark. The construction from the theorem is functorial. In particular, the semigroup of linear relations on $W \times W$ (to be more precise: a quotient with respect to scalars) acts by structural pairs on $\mathcal{X}$.

Theorem 2.7. *We define* operators of left-, middle- and right multiplication on $\mathcal{X}$ *by*

$$L_{xayb}(z) := R_{aybz}(x) := M_{xabz}(y) := \Gamma(x,a,y,b,z).$$

Then, for all $x,a,y,b,z \in \mathcal{X}$*, the pairs*

$$(L_{xayb}, L_{yaxb}), \quad (M_{xabz}, M_{zabx}), \quad (R_{aybz}, R_{azby})$$

are structural transformations of the Grassmannian geometry $\mathcal{X}$*.*

Proof. Let $\mathbf{l}_{x,a,y,b} \subset W \oplus W$ be the linear relation defined by

$$\mathbf{l}_{x,a,y,b} := \{(\zeta,\omega) \in W \oplus W \mid \exists \xi \in x : \omega + \zeta \in a, \omega + \zeta + \xi \in y, \omega + \xi \in b\}.$$

Then it follows immediately by using the (x,z)-description that

$$(\mathbf{l}_{x,a,y,b})_*(z) = \{\omega \in W \mid \exists \zeta \in z : (\zeta,\omega) \in \mathbf{l}_{x,a,y,b}\} = \Gamma(x,a,y,b,z) = L_{xayb}(z).$$

On the other hand,

$$\begin{aligned}(\mathbf{l}_{x,a,y,b})^*(z) &= \{\omega \in W \mid \exists \zeta \in z : (\omega,\zeta) \in \mathbf{l}_{x,a,y,b}\} \\ &= \Big\{\omega \in W \,\Big|\, \exists \zeta \in z, \exists \xi \in x : \omega + \zeta \in a, \omega + \zeta + \xi \in y, \zeta + \xi \in b\Big\} \\ &= \Gamma(y,a,x,b,z) = L_{yaxb}(z)\,,\end{aligned}$$

where the third equality follows by using the (y,z)-description with permuted variables. This proves that (L_{xayb}, L_{yaxb}) is a structural pair; the claim for right multiplications is just an equivalent version of this, and the claim for middle multiplications is proved in the same way as above. □

Remark. We have proved that, in terms of inverses of linear relations,

$$(\mathbf{l}_{x,a,y,b})^{-1} = \mathbf{l}_{y,a,x,b}. \tag{2.7}$$

If $x \top a$ and $y \top b$, then $\mathbf{l}_{xaby}$ is the graph of the linear operator $L_{xayb} \in \mathrm{End}(W)$; for $x,y \in U_{ab}$, this operator is invertible and the preceding formula holds in the sense of an operator equation.

Corollary 2.8. *The multiplication map satisfies the following "self-distributivity" identities:*

$$\Gamma\Big(x,a,\Gamma\big(u,\Gamma(a,z,c,x,b),v,\Gamma(a,z,d,x,b),w\big),b,z\Big) = \\ \Gamma\Big(\Gamma(x,a,u,b,z),c,\Gamma(x,a,v,b,z),d,\Gamma(x,a,w,b,z)\Big)$$

$$\Gamma\Big(x,a,y,b,\Gamma\big(u,\Gamma(y,a,x,b,c),v,\Gamma(y,a,x,b,d),w\big)\Big) = \\ \Gamma\Big(\Gamma(x,a,y,b,u),c,\Gamma(x,a,y,b,v),d,\Gamma(x,a,y,b,w)\Big)$$

Proof. The first identity follows by applying the adjoint pair $(f,f^t) = (M_{xabz}, M_{zabx})$ to $\Gamma(u,c,v,d,w)$ (and using the symmetry property), and similarly the second by using the pair $(f,f^t) = (L_{xayb}, L_{yaxb})$. □

Corollary 2.9. *For all $a, b \in \mathcal{X}$, the maps $(\ \ )^+ : U_a \times U_b \times U_a \to U_a$ and $(\ \ )^- : U_b \times U_a \times U_b \to U_b$ defined in Theorem 1.5 are tri-affine (i.e., affine in all three variables) and satisfy the para-associative law*

$$(xy(uvw)^\pm)^\pm = ((xyu)^\pm vw)^\pm = (x(vuy)^\mp w)^\pm .$$

Proof. Let us show that M_{xabz} induces an affine map $U_b \to U_a$, $y \mapsto (xyz)^+$, for fixed $x, z \in U_a$. We know already that this map is well-defined (Theorem 1.5). Since $(f, g) = (M_{xabz}, M_{zabx})$ is structural, the map $f : U_{g(a)} \to U_a$ is affine, where (according to Corollary 2.5, (1)),

$$g(a) = M_{zabx}(a) = \Gamma(z, a, a, b, x) = \Gamma(a, z, a, x, b) = b.$$

By the same kind of argument, using Corollary 2.5, (2) and (3), wee see that the other partial maps are affine. The corresponding statements for $(\ \ )^-$ follow by symmetry, and the para-associative law follows by restriction of the para-associative law in the semi-groud $\mathcal{X}_{ab}$. □

Remark. For $a = b$, we get the additive groud U_a, and if $U_{ab} \neq \emptyset$, then we get a sort of "triaffine extension" of the groud U_{ab}. If $a \top b$, then we have base points a in U_b and b in U_a, and obtain a trilinear product (Theorem 1.5).

2.6. **The extended dilation map.** Next we (re-)define, for $r \in \mathbb{K}$, the *dilation map* $\Pi_r : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ by the following equivalent expressions

$$\begin{aligned}
\Pi_r(x, a, z) &:= \Big\{ \omega \in W \,\Big|\, \exists \alpha \in a, \exists \zeta \in z, \exists \xi \in x : \omega - r\alpha = \xi = \zeta - \alpha \Big\} \\
&= \Big\{ \omega \in W \,\Big|\, \exists \alpha \in a, \exists \zeta \in z, \exists \xi \in x : \omega + (1-r)\alpha = \zeta = \alpha + \xi \Big\} \\
&= \Big\{ \omega \in W \,\Big|\, \exists \alpha \in a, \exists \zeta \in z, \exists \xi \in x : \omega = (1-r)\xi + r\zeta,\ \zeta - \xi = \alpha \Big\}
\end{aligned}$$

We refer to the last expression as the "(x, z)-description", and we define partial maps $\mathcal{X} \to \mathcal{X}$ by

$$\lambda^r_{xa}(z) := \rho^r_{az}(x) := \mu^r_{xz}(a) := \Pi_r(x, a, z)$$

(where λ reminds us of "left", ρ "right" and μ "middle").

Theorem 2.10. *The map $\Pi_r : \mathcal{X}^3 \to \mathcal{X}$ extends the ternary map defined in the preceding chapter (and denoted by the same symbol there), and it has the following properties:*

(*1*) Symmetry: *$\mu^r_{xz} = \mu^{1-r}_{zx}$, that is, $\lambda^r_{xa} = \rho^{1-r}_{ax}$ or*

$$\Pi_r(x, a, z) = \Pi_{1-r}(z, a, x).$$

(*2*) Multiplicativity: *if $x \top a$ and $r, s \in \mathbb{K}$,*

$$\Pi_r(x, a, \Pi_s(x, a, y)) = \Pi_{rs}(x, a, y),$$

(*3*) Diagonal values:

$$\Pi_r(x, a, x) = x, \quad \Pi_0(x, a, z) = \Pi_1(z, a, x) = x \wedge (z \vee a) = \Gamma(a, x, x, a, z).$$

(*4*) Structurality: *if $r(1-r) \in \mathbb{K}^\times$, then, for all $x, a, z \in \mathcal{X}$, the pairs*

$$(\lambda^r_{xa}, \lambda^r_{ax}), \quad (\mu^r_{xz}, \mu^r_{zx})$$

are structural transformations of $(\mathcal{X}, \Gamma)$.

Proof. The symmetry relation (1) follows directly from the (x, z)-description.

Next we show that Π_r coincides with the dilation map from the preceding chapter. Assume first that $x \top a$. We show that $\Pi_r(x, a, z) = (rP^x_a + P^a_x)(z)$:

$$\begin{aligned}(rP^x_a + P^a_x)(z) &= \Big\{ e \in W \,\Big|\, \exists \zeta \in z : e = rP^x_a(\zeta) + P^a_x(\zeta) \Big\} \\ &= \Big\{ e \in W \,\Big|\, \exists \alpha \in a, \exists \zeta \in z, \exists \xi \in x : e - r\alpha = \zeta - \alpha = \xi \Big\} = \Pi_r(x, a, z)\end{aligned}$$

writing $\zeta = P^x_a(\zeta) + P^a_x(\zeta) = \alpha + \xi$. For $z \top a$, the claim follows now from the symmetry relation (1).

(3) With $\omega = (1 - r)\xi + r\zeta$, it follows for $x = z$ that $\Pi_r(x, a, x) \subset x$. Conversely, we get $x \subset \Pi_r(x, a, x)$ by letting $\alpha = 0$ and $\zeta = \xi$, given $\xi \in x$. The other relations are proved similarly.

(2) Under the assumption $x \top a$, the claim amounts to the operator identity

$$(rP^x_a + P^a_x)(sP^x_a + P^a_x) = (rsP^x_a + P^a_x)$$

which is easily checked.

(4) Fix $x, a \in \mathcal{X}$, $r \in \mathbb{K}$ and define the linear subspace $\mathbf{r} \subset W \oplus W$ by

$$\mathbf{r} := \mathbf{r}_{xa} := \big\{ (\zeta, \omega) \in W \oplus W \,|\, \exists \alpha \in a, \exists \xi \in x : \ \omega = \zeta - (1 - r)\alpha, \ \zeta - \alpha = x \big\}$$

Then

$$\mathbf{r}_*(z) = \{ \omega \in W \,|\, \exists \zeta \in z : (\zeta, \omega) \in \mathbf{r} \} = \Pi_r(x, a, z).$$

On the other hand, by a straightforward change of variables (which is bijective since r is assumed to be invertible), one checks that

$$\mathbf{r}^*(z) = \{ \omega \in W \mid \exists \zeta \in z : (\omega, \zeta) \in \mathbf{r} \} = \Pi_r(a, x, z).$$

Hence $(\lambda^r_{xa}, \lambda^r_{ax}) = (\mathbf{r}_*, \mathbf{r}^*)$ is structural. The calculation for the middle multiplications is similar. □

Remark. If $x \top a$ and $r \in \mathbb{K}$ an arbitrary scalar, we still have structurality in (4). The situation is less clear if x, a, r are all arbitrary.

Remark. One can define *structurality with respect to* Π_r in the same way as for Γ, by conditions of the form

$$f\big(\Pi_r(x, g(a'), z\big) = \Pi'_r(f(x), a', f(z)), \quad g\big(\Pi'_r(x', f(a), z'\big) = \Pi_r(g(x'), a, g(z')).$$

Then partial maps of Γ are structural for Π_r, and partial maps of Π_s are structural for Π_r (this property has been used in [Be02] to characterize *generalized projective geometries*). The proofs are similar to the ones given above.

3. Associative geometries

In this chapter we give an axiomatic definition of associative geometry, and we show that, at a base point, the corresponding "tangent object" is an associative pair. Conversely, given an associative pair, one can reconstruct an associative geometry. The question whether these constructions can be refined to give a suitable equivalence of categories will be left for future work.

3.1. **Axiomatics.**

Definition. *An* associative geometry *over a commutative unital ring* $\mathbb{K}$ *is given by a set* $\mathcal{X}$ *which carries the following structures:* $\mathcal{X}$ *is a* complete lattice *(with join denoted by* $x \vee y$ *and meet denoted by* $x \wedge y$*), and maps (where* $s \in \mathbb{K}$*)*

$$\Gamma : \mathcal{X}^5 \to \mathcal{X}, \quad \Pi_s : \mathcal{X}^3 \to \mathcal{X},$$

such that the following holds. We use the notation

$$L_{xaby}(z) := M_{xabz}(y) := R_{aybz}(x) := \Gamma(x, a, y, b, z)$$

for the partial maps of Γ*, and call* x *and* y transversal*, denoted by* $x\top y$*, if* $x \vee y = 0$ *and* $x \wedge y = 1$*, and we let*

$$C_a := a^\top := \{x \in \mathcal{X} \mid x\top a\}, \quad C_{ab} := C_a \cap C_b$$

for sets of elements transversal to a*, resp. to* a *and* b*.*

(*1*) The semigroud property: *for all* $x, y, z, u, v, a, b \in \mathcal{X}$*:*

$$\Gamma(\Gamma(x, a, y, b, z), a, u, b, v) = \Gamma(x, a, \Gamma(u, a, z, b, y), b, v) = \Gamma(x, a, y, b, \Gamma(z, a, u, b, v)).$$

In other words, for fixed a, b*, the product* $(xyz) := \Gamma(x, a, y, b, z)$ *turns* $\mathcal{X}$ *into a semigroud, which will be denoted by* $\mathcal{X}_{ab}$*.*

(*2*) Invariance of Γ under the Klein 4-group in (x, a, b, z): *for all* $x, a, y, b, z \in \mathcal{X}$*,*
  (*i*) $\Gamma(x, a, y, b, z) = \Gamma(z, b, y, a, x)$
  (*ii*) $\Gamma(x, a, y, b, z) = \Gamma(a, x, y, z, b)$

  In particular, $\mathcal{X}_{ba}$ *is the opposite semigroud of* $\mathcal{X}_{ab}$*.*

(*3*) Structurality of partial maps: *for all* $x, a, y, b, z \in \mathcal{X}$*, the pairs*

$$(L_{xayb}, L_{yaxb}), \quad (M_{xabz}, M_{zabx}), \quad (R_{aybz}, R_{azby})$$

are structural transformations (see definition below).

(*4*) Diagonal values:
  (*i*) *for all* $a, b, y \in \mathcal{X}$*,* $\Gamma(a, a, y, b, b) = a \vee b$*,*
  (*ii*) *for all* $a, b, y \in \mathcal{X}$*,* $\Gamma(a, b, y, a, b) = a \wedge b$*,*
  (*iii*) *if* $x \in C_{ab}$*, then* $\Gamma(x, a, x, b, z) = z = \Gamma(z, b, x, a, x)$*,*
  (*iv*) *if* $a\top x$ *and* $y\top b$*, then* $\Gamma(x, a, y, b, b) = b$*,*
  (*v*) *if* $a\top y$ *and* $b\top x$*, then* $\Gamma(x, a, y, b, a) = a$*.*

(*5*) The affine space property: *for all* $a \in \mathcal{X}$ *and* $r \in \mathbb{K}$*,* C_a *is stable under the dilation map* Π_r*, and* C_a *becomes an affine space with additive groud structure*

$$x - y + x = \Gamma(x, a, y, a, z)$$

and scalar action given for $x, y \in C_a$ *by*

$$r \cdot_x y = (1 - r)x + ry = \Pi_r(x, a, y).$$

(*6*) The semigrouded pairs: *for all* $a, b \in \mathcal{X}$*,*

$$\Gamma(U_a, a, U_b, b, U_a) \subset U_a, \quad \Gamma(U_b, a, U_a, b, U_b) \subset U_b.$$

Definition. *The* opposite geometry *of an associative geometry* $(\mathcal{X}, \top, \Gamma, \Pi)$*, denoted by* $\mathcal{X}^{op}$*, is* $\mathcal{X}$ *with the same transversality relation* $\top$*, the same dilation map* Π *and the opposite pentary product map*

$$\Gamma^{op}(x, a, y, b, z) := \Gamma(z, a, y, b, x).$$

A base point *in $\mathcal{X}$ is a fixed transversal pair (o^+, o^-), and the* dual base point *in $\mathcal{X}$ is then (o^-, o^+).*

Definition. Homomorphisms of associative geometries *are maps $\phi : \mathcal{X} \to \mathcal{Y}$ such that*

$$\phi\big(\Gamma(x,a,y,b,z)\big) = \Gamma(\phi x, \phi a, \phi y, \phi b, \phi z)$$
$$\phi\big(\Pi_r(x,a,y)) = \Pi_r(\phi x, \phi a, \phi y)\big)$$

It is clear that associative geometries over $\mathbb{K}$ with their homomorphisms form a category. Antihomomorphisms *are homomorphisms into the opposite geometry.* Involutions *are antiautomorphims of order two; they play an important rôle which will be discussed in subsequent work* ([BeKi09]). *For a fixed base point (o^+, o^-), we define the* structure group *as the group of automorphisms of $\mathcal{X}$ that preserve (o^+, o^-). Note that the structure group acts linearly on (V^+, V^-).*

An adjoint *or* structural pair of transformations *is a pair $g : \mathcal{X} \to \mathcal{Y}$, $h : \mathcal{Y} \to \mathcal{X}$ such that*

$$g\big(\Gamma(h(x), u, h(y), v, h(z))\big) = \Gamma(x, g(u), y, g(v), z)\,,$$
$$g\big(\Pi_r(h(x), u, h(y))\big) = \Pi_r(x, g(u), y)$$

and vice versa. Clearly, this also defines a category.

3.2. **Consequences.** We are going to derive some easy consequences of the axioms.

Lemma 3.1. *Isomorphisms of associative geometries preserve the lattice structure, and antiautomorphisms reverse the lattice structure.*

Proof. This follows readily from the Diagonal Value Axiom (4), (i), (ii). □

Next, let us rewrite the semigroud property in operator form:

$$\begin{array}{ccccc} R_{aubv}L_{xayb} & = & M_{xabv}M_{uaby} & = & L_{xayb}R_{aubv} \\ L_{xayb}L_{zaub} & = & L_{x,a,L_{ybza}(u),b} & = & L_{L_{xayb}(z),a,y,b} \\ M_{\Gamma(x,a,y,b,z),a,b,v} & = & M_{xabv}L_{ybza} & = & L_{xayb}M_{zabv} \end{array}$$

Assume that $x, y \in U_{ab}$ and $z \in \mathcal{X}$. Then, by Axiom (4) (iii), $L_{xaxb} = \mathrm{id}_{\mathcal{X}} = L_{ybyb}$, whence $L_{xayb}L_{yaxb} = L_{x,a,L_{ybyb}(x),b} = L_{xaxb} = \mathrm{id}_{\mathcal{X}}$. Thus $L_{xayb} : \mathcal{X} \to \mathcal{X}$ is invertible with inverse

$$(L_{xayb})^{-1} = L_{yaxb}.$$

By (2), this is equivalent to $(R_{aybx})^{-1} = R_{axby}$, and in the same way one shows that M_{xaby} is invertible with inverse

$$(M_{xaby})^{-1} = M_{xbay}.$$

It follows that L_{xayb}, R_{aybx} and M_{xaby} are automorphisms of the geometry. In particular, M_{xaay} and M_{xabx} are automorphisms of order two.

Proposition 3.2. *For all $a, b \in \mathcal{X}$, C_{ab} is stable under the ternary map $(x,y,z) \mapsto \Gamma(x,a,y,b,z)$, which turns it into a groud denoted by U_{ab}. For any $y \in U_{ab}$, the group (U_{ab}, y) acts on $\mathcal{X}$ from the left and from the right by the formulas given in Theorem 1.3, and both actions commute.*

Proof. As remarked above, L_{xayb} is an automorphism of the geometry. It stabilizes a and b and hence also C_a and C_b. Thus C_{ab} is stable under the ternary map, and the para-associative law and the idempotent law hold by Axioms (1) and (4) (iii). The remaining statements follow easily from Axiom (1). $\square$

In Part II ([BeKi09]) we will also describe the "Lie algebra" of U_{ab}, thus giving a relatively simple description of the group structure of U_{ab}. – Next we give the promised conceptual interpretation of Equation (1.5).

Lemma 3.3. *For all $z \in U_b$, $x \in U_{ab}$, and all $y \in \mathcal{X}$,*

$$\Gamma\big(x, b, \Gamma(x, a, y, b, z), b, z\big) = \Gamma(z, b, a, y, x).$$

Proof. Using that $R_{xaxb} = \mathrm{id}_{\mathcal{X}}$ for $a, b \in U_x$, we have, for all $x, z \in U_b$,

$$\begin{aligned} \Gamma\big(x, b, \Gamma(x, a, y, b, z), b, z\big) &= M_{xbbz} M_{xabz}(y) \\ &= M_{bzxb} M_{bzxa}(y) \\ &= L_{bzax} R_{zbxb}(y) \\ &= L_{bzax}(y) = \Gamma(b, z, a, x, y) = \Gamma(z, b, a, y, x). \end{aligned}$$

$\square$

Since the operator M_{xbbz} is invertible with inverse M_{zbbx}, we have, equivalently,

$$\Gamma(x, a, y, b, z) = \Gamma(z, b, \Gamma(z, b, a, y, x), b, x).$$

If a and b are transversal, we may rewrite the lemma in the form of Equation (1.5): with $b = o^-$, $y = o^+$: for all $x, z \in V^+$,

$$\Gamma(z, o^-, a, o^+, x) = \Gamma(x, o^-, \Gamma(x, a, o^+, o^-, z), o^-, z) = x - \Gamma(x, a, o^+, o^-, z) + z.$$

We will see in the following result that $\Gamma(z, o^-, a, o^+, x)$ is trilinear in (z, a, x), and hence $\Gamma(x, a, o^+, o^-, z)$ is tri-affine in (x, a, z). Both expressions can be considered as geometric interpretations of the associative pair attached to (o^+, o^-). More generally, the lemma implies the following analog of Axiom (6): *for all $b, y \in \mathcal{X}$ (transversal or not), the map*

$$U_b \times U_y \times U_b \to U_b, \quad (x, a, z) \mapsto \Gamma(x, a, y, b, z)$$

is well-defined and affine in all three variables.

3.3. **From geometries to associative pairs.**

Theorem 3.4. *Let $(\mathcal{X}, \top, \Gamma, \Pi_r)$ be an associative geometry over $\mathbb{K}$.*

i) Assume $\mathcal{X}$ admits a transversal pair, which we take as base point (o^+, o^-). Then, letting $\mathbb{A}^+ := U_{o^-}$ and $\mathbb{A}^- := U_{o^+}$, the pair of linear spaces $(\mathbb{A}^+, \mathbb{A}^-)$ with origins o^+, resp. o^-, becomes an associative pair when equipped with

$$< xbz >^+ := \Gamma(x, o^-, b, o^+, z), \quad < ayc >^- := \Gamma(a, o^-, y, o^+, c).$$

This construction is functorial (in the "usual" category).

ii) Assume $\mathcal{X}$ admits a transversal triple (a, b, c). Then, letting $\mathbb{B} := U_c$, the $\mathbb{K}$-module $\mathbb{B}$ with origin $o^+ := a$ becomes an associative unital algebra with unit $u := b$ and product map

$$\mathbb{A} \times \mathbb{A} \to \mathbb{A}, \quad (x, z) \mapsto xz := \Gamma(x, a, u, c, z).$$

This construction is functorial (in the "usual" category).

Proof. (i) By the "semi-grouded pair axiom" (6), the maps $\mathbb{A}^\pm \times \mathbb{A}^\mp \times \mathbb{A}^\pm \to \mathbb{A}^\pm$ are well-defined. By restriction from $\mathcal{X}_{o^+,o^-}$, they satisfy the para-associative law. They are tri-affine: the proof is exactly the same as the one of Corollary 2.9. Thus it only remains to be shown that they are trilinear, with respect to the origins $o^\pm \in \mathbb{A}^\pm$. Let $x, z \in \mathbb{A}^+$ and $b \in \mathbb{A}^-$. Then

$$< xbo^+ >^+ = \Gamma(x, o^-, b, o^+, o^+) = o^+, \quad < o^+bz >^+ = \Gamma(o^+, o^-, b, o^+, z) = o^+$$
$$< xo^-z >^+ = \Gamma(x, o^-, o^-, o^+, z) = \Gamma(o^-, x, o^-, z, o^+) = o^+.$$

by the Diagonal Value Axiom (4). If $\phi : \mathcal{X} \to \mathcal{Y}$ is a base-point preserving homomorphism, then restriction of ϕ yields, by definition of a homomorphism, a pair of $\mathbb{K}$-linear maps $\mathbb{A}^\pm \to (\mathbb{A}')^\pm$, which commutes with the product maps Γ, Γ' and hence is a homomorphism of associative pairs.

(ii) With notation from (i), we have $xz =< xuz >^+$, and hence the product is well-defined, bilinear and associative $\mathbb{A} \times \mathbb{A} \to \mathbb{A}$. We only have to show that u is a unit element: but this is immediate from $xu = \Gamma(x, a, u, b, u) = x = \Gamma(u, a, u, b, x) = ux$. □

Example. For any $\mathbb{B}$-module W, the Grassmannian geometry $\mathcal{X}$ is an associative geometry, by the results of Chapter 2. For a decomposition $W = o^+ \oplus o^-$, the corresponding associative pair is $(\mathbb{A}^+, \mathbb{A}^-) = (\mathrm{Hom}_{\mathbb{B}}(o^+, o^-), \mathrm{Hom}_{\mathbb{B}}(o^-, o^+))$, by Theorem 1.7. In case W is a topological module over a topological ring $\mathbb{K}$, we may also work with subgeometries of the whole Grassmannian, such as Grassmannians of *closed subspaces with closed complement.* For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, if W is, e.g., a Banach space, the associated associative pair is a *pair of spaces of bounded linear operators.*

Remark. There is a natural definition of *structural transformations of associative pairs.* They are induced by structural pairs (f, g) satisfying $f(o^+) = o^+$, $g(o^-) = o^-$ and $f(\mathbb{A}^+) \subset \mathbb{A}^+$, $g(\mathbb{A}^-) \subset \mathbb{A}^-$. With respect to such pairs, the construction from the theory is still functorial.

3.4. **From associative pairs to geometries.**

Theorem 3.5. *i) For every associative pair* $(\mathbb{A}^+, \mathbb{A}^-)$ *there exists an associative geometry* $\mathcal{X}$ *with base point* (o^+, o^-) *having* $(\mathbb{A}^+, \mathbb{A}^-)$ *as associated pair.*
ii) For every unital associative algebra $(\mathbb{A}, 1)$ *there exists an associative geometry* $\mathcal{X}$ *with transversal triple* (o^+, Δ, o^-) *having* $(\mathbb{A}, 1)$ *as associated algebra.*

Proof. (ii) Let $W = \mathbb{A} \oplus \mathbb{A}$, o^+ the first and o^- the second factor and Δ the diagonal. Then (o^+, Δ, o^-) is a transversal triple in the Grassmannian geometry $\mathcal{X} = \mathrm{Gras}_{\mathbb{A}}(W)$, and its associated algebra is $\mathbb{A} \cong \mathrm{Hom}_{\mathbb{A}}(\mathbb{A}, \mathbb{A})$ (see the preceding example, with $o^+ \cong o^- \cong \mathbb{A}$). Note that the connected component of o^+ can be interpreted as the *projective line over* $\mathbb{A}$, cf. [BeNe05], [Be08].

(i) Consider any algebra imbedding $(\hat{\mathbb{A}}, e)$ of the pair $(\mathbb{A}^+, \mathbb{A}^-)$, for instance, its standard imbedding (see Appendix B). Let $\mathcal{X} = \mathrm{Gras}_{\hat{\mathbb{A}}}(\hat{\mathbb{A}})$ be the Grassmannian of all *right* ideals in $\hat{\mathbb{A}}$. As base point in $\mathcal{X}$ we choose

$$o^+ := e\hat{\mathbb{A}} = \mathbb{A}_{11} \oplus \mathbb{A}_{10}, \quad o^- := f\hat{\mathbb{A}} = \mathbb{A}_{00} \oplus \mathbb{A}_{01}.$$

where $f = 1 - e$. Recall that $\mathbb{A}^+ = \mathbb{A}_{10} = e\hat{\mathbb{A}}f$ and $\mathbb{A}^- = \mathbb{A}_{01} = f\hat{\mathbb{A}}e$. The associative pair corresponding to $(\mathcal{X}; o^+, o^-)$ is (see example at the end of the preceding section)

$$(\mathrm{Hom}_{\hat{\mathbb{A}}}(o^-, o^+), \mathrm{Hom}_{\hat{\mathbb{A}}}(o^+, o^-)).$$

But this pair is naturally isomorphic to $(\mathbb{A}^+, \mathbb{A}^-)$. Indeed,

$$\mathrm{Hom}_{\hat{\mathbb{A}}}(e\hat{\mathbb{A}}, f\hat{\mathbb{A}}) \to \mathbb{A}_{01} = f\hat{\mathbb{A}}e, \quad g \mapsto g(e)$$

is $\mathbb{K}$-linear, well-defined (since $g(e)e = g(ee) = g(e)$, g being a right $\hat{\mathbb{A}}$-linear map) and has as inverse mapping $c \mapsto (x \mapsto cx)$, hence is a $\mathbb{K}$-isomorphism. Identifying both pairs of $\mathbb{K}$-modules in this way, a direct check shows that the triple products also coincide, thus establishing the desired isomorphism of associative pairs. □

Remark. It is of course also possible to see (ii) as a special case of (i). In this case we may work with the algebra imbedding of $(\mathbb{A}, \mathbb{A})$ into the matrix algebra $\hat{\mathbb{A}} = M(2, 2; \mathbb{A})$, *cf.* Appendix B.

Remark (**Functoriality**). Is the construction from the preceding theorem functorial, or can it be modified such that it becomes functorial? If this is possible, do we get an equivalence of categories between associative pairs and certain associative geometries with base point (whose algebraic properties reflect "connectedness and simply connectedness")? Since the construction of the standard imbedding is not functorial (see, however, [Ca04]), the answer is not clear. Nevertheless, one might be optimistic here, since results of a similar kind hold indeed in a Jordan theoretic context (see [Be02]).

4. Further topics

(1) Jordan geometries revisited. The present work sheds new light on geometries associated to *Jordan algebraic structures*: in the same way as associative pairs give rise to Jordan pairs by restricting to the diagonal ($Q(x)y =< xyx >$; see Appendix B), associative geometries give rise to "Jordan geometries". The new feature is that we get *two* diagonal restrictions $\Gamma(x, a, y, b, x)$ and $\Gamma(x, a, y, a, z)$ which are equivalent. They can be used to give a new axiomatic foundation of "Jordan geometries". Unlike the theory developed in [Be02], this new foundation will be valid also in case of characteristic 2 and hence corresponds to general quadratic Jordan pairs. In this theory, the grouds from the associative theory will be replaced by *symmetric spaces* (the diagonal (xyx)).

(2) Involutions, Jordan-Lie algebras, classical groups. From a Lie theoretic point of view, the present work deals with classical groups of type A_n (the "general linear" family). The other classical series (orthogonal, unitary and symplectic families) can be dealt with by adding an *involution* to an associative geometry. This will be discussed in detail in [BeKi09]. From a more algebraic point of view, this amounts to look at *Jordan-Lie* or *Lie-Jordan algebras* instead of associative pairs (and hence is closely related to (1)), and to ask for the geometric counterpart. In [Be08], it is advocated that this might also be interesting in relation with foundational issus of quantum mechanics.

(3) Tensor Products. In the associative and in the Jordan-Lie categories, *tensor products* exist (cf. [Be08] for historical remarks on this item in relation with foundations of Quantum Mechanics). What is the geometric interpretation of this remarkable fact?

(4) Alternative Geometries. The geometric object corresponding to *alternative pairs* (see [Lo75]) should be a collection of Moufang loops, interacting among each other in a similar way as the grouds U_{ab} do in an associative geometry.

(5) Classical projective geometry revisted. The grouds U_{ab} show already up in ordinary projective spaces, and their alternative analogs will show up in octonion projective planes. It should be interesting to review classical approaches from this point of view.

(6) Invariant Theory. The problem of classifying the grouds U_{ab} in a given geometry $\mathcal{X}$ is very close to classifying orbits in $\mathcal{X} \times \mathcal{X}$ under the automorphism group. Invariants of grouds ("rank") give rise to invariants of pairs. Similarly, invariants of groups (U_{ab}, y) give rise to invariants of triples ("rank and signature"), and invariants (conjugation class) of projective endomorphisms L_{xayb} to invarinats of quadrupels ("cross-ratio").

(7) Structure theory: ideals and intrinsic subspaces. We ask to translate features of the structure theory of associative pairs and algebras to the level of associative geometries: what are the geometric notions corresponding to left-, right- and inner ideals? See [BeL08] for the Jordan case.

(8) Positivity and convexity: case of C^*-algebras. C^*-algebras and related triple systems ("ternary rings of operators", see [BM04]) are distinguished among general ones by properties involving "positivity" and "convexity". What is their geometric counterpart on the level of associative geometries? Note that these properties are really ones of the involution $*$, so these questions can be seen to fall in the realm of Topic (2).

Appendix A: grouds and semigrouds

Definition. *A* groud $(G, (\cdot,\cdot,\cdot))$ *is a set G together with a ternary operation* $G^3 \to G; (x,y,z) \mapsto (xyz)$ *satisfying the identities (G1) and (G2) discussed in the Introduction (§0.2).*

The term has been suggested by Boris Schein in various publications (*e.g.*, [Sch62]) as a substitute for the term "heap", which has been used for the same notion (as have "flock" and "herd"). An early term, due to Prüfer was *Schar*, which was translated by Suschkewitsch into Russian as груд. This was later somewhat unfortunately translated into English as "heap". Schein has suggested adapting the Russian term directly into English as *groud*, and we follow that suggestion here. For more on the history of the concept, as well as of semigrouds defined below, we refer the reader to the work of B. Schein, *e.g.*, [Sch62].

In a groud $(G, (\cdot,\cdot,\cdot)$, introduce *left-*, *right-* and *middle multiplications* by

$$(xyz) =: \ell_{x,y}(z) =: r_{y,z}(x) =: m_{x,z}(y)\,.$$

Then the axioms of a groud can be rephrased thusly:

(G1') $$\ell_{x,y} \circ r_{u,v} = r_{u,v} \circ \ell_{x,y}$$

(G2') $$\ell_{x,x} = r_{x,x} = \mathrm{id}$$

or, in yet another way,

(G1") $$\ell_{x,y} \circ \ell_{z,u} = \ell_{\ell_{x,y}(z),u}$$

(G2") $$\ell_{x,y}(y) = r_{y,x}(y) = x\,.$$

Taking $y = z$ in (G1"), and using (G2"), we get what one might call "Chasle's relation" for left translations

$$\ell_{x,y} \circ \ell_{y,u} = \ell_{x,u}$$

which for $u = x$ shows that the inverse of $\ell_{x,y}$ is $\ell_{y,x}$. Similarly, we have a Chasle's relation for right translations, and the inverse of $r_{x,y}$ is $r_{y,x}$. Unusual, compared to group theory, is the rôle of the middle multiplications. Namely, fixing for the moment a unit e, we have

$$(x(uyw)z) = x(uy^{-1}w)^{-1}z = xw^{-1}yu^{-1}z = ((xwy)uz) = (xw(yuz))$$

(the *para-associative law*, cf. relation (G3), Introduction), i.e.,

(G3') $$m_{x,z} \circ m_{u,w} = \ell_{x,w} \circ r_{u,z} = r_{u,z} \circ \ell_{x,w}.$$

Taking $x = w$, resp. $u = z$, we see that all left and right multiplications can be expressed via middle multiplications:

$$r_{u,z} = m_{x,z} \circ m_{u,x}, \quad \ell_{x,w} = m_{x,z} \circ m_{z,w}.$$

Taking $u = z$, resp. $x = w$, we see that $m_{x,z} \circ m_{z,x} = \mathrm{id}$, hence middle multiplications are invertible. In particular $m_{x,x}^2 = \mathrm{id}$, which reflects the fact that $m_{x,x}$ is inversion in the group (G, x). Also, (G3') implies that

$$m_{x,e} \circ m_{x,e} = r_{x,e} \circ \ell_{x,e} = \ell_{x,e} \circ (r_{e,x})^{-1},$$

which means that conjugation by x in the group with unit e is equal to $(m_{x,e})^2$.

Since a groud can be viewed as an equational class in the sense of universal algebra $(G, (\cdot,\cdot,\cdot))$, all of the usual notions apply. For instance, a *homomorphism of grouds* is a map $\phi : G \to H$ such that $\phi\big((xyz)\big) = (\phi(x)\phi(y)\phi(z))$, and an *anti-homomorphism of grouds* is a homomorphism to the *opposite groud* (same set with product $(x, y, z) \mapsto (zyx)$). Homomorphisms enjoy similar properties as usual affine maps. It is easily proved that left and right multiplications are automorphisms (called *inner*), whereas middle multiplications are inner anti-automorphisms. Other notions, such as subgrouds, products, congruences and quotients follow standard patterns.

Definition. *A* semigroud $(G, (\cdot,\cdot,\cdot))$ *is a set G with a ternary operation $G^3 \to G; (x, y, z) \mapsto (xyz)$ satisfying the para-associative law (G3) from the Introduction.*

The basic example is the *symmetric semigroud* on sets A and B, the set of all relations between A and B with $(rst) = r \circ s^{-1} \circ t$, where $\circ$ is the composition of relations.

Clearly, fixing the middle element in a semigroud gives rise to a semigroup; but, in contrast to the case of groups, not all semigroups are obtained in this way. For more on semigrouds, see, *e.g.* [Sch62] and the references therein.

Appendix B: Associative pairs

Definition. An *associative pair (over $\mathbb{K}$)* is a pair $(\mathbb{A}^+, \mathbb{A}^-)$ of $\mathbb{K}$-modules together with two trilinear maps

$$< \cdot, \cdot, \cdot >^{\pm} : \mathbb{A}^{\pm} \times \mathbb{A}^{\mp} \times \mathbb{A}^{\mp} \to \mathbb{A}^{\pm}$$

such that

$$< xy < zuv >^{\pm} >^{\pm} = << xyz >^{\pm} uv >^{\pm} = < x < uzy >^{\mp} v >^{\pm} .$$

Note that we follow here the convention of Loos [Lo75]. Other authors (e.g. [MMG]) use a modified identity, replacing the last term by $< x < yzu >^{\mp} v >^{\pm}$. But both versions are equivalent: it suffices to replace $< \ >^-$ by the trilinear map $(x, y, z) \mapsto < z, y, x >^-$. We prefer the definition given by Loos since it takes the same form as the para-associative law in a semigroud. We should mention, however, that for *associative triple systems*, i.e., $\mathbb{K}$-modules $\mathbb{A}$ with a trilinear map $\mathbb{A}^3 \to \mathbb{A}$, $(x, y, z) \mapsto < xyz >$ these two versions of the defining identity have to be distinguished, leading to two different kinds of associative triple systems ("ternary rings", cf. [Li71], and associative triple systems [Lo72]; all this is best discussed in the context of associative pairs, resp. geometries, *with involution*, see Topic (2) in Chapter 4 and [BeKi09].) In any case, for fixed $a \in \mathbb{A}^-$, $\mathbb{A}^+$ with

$$x \cdot_a y := < xay >$$

is an associative algebra, called the *a-homotope* and denoted by $\mathbb{A}^+_a$.

Examples of associative pairs.

(1) Every associative algebra $\mathbb{A}$ gives rise to an associative pair $\mathbb{A}^+ = \mathbb{A}^- = \mathbb{A}$ via $< xyz >^+ = xyz$, $< xyz >^- = zyx$.
(2) For $\mathbb{K}$-modules E and F, let $\mathbb{A}^+ = \mathrm{Hom}(E, F)$, $\mathbb{A}^- = \mathrm{Hom}(F, E)$,
$$< XYZ >^+ = X \circ Y \circ Z \qquad < XYZ >^- = Z \circ Y \circ X.$$
(3) Let $\hat{\mathbb{A}}$ be an associative algebra with unit 1 and idempotent e and $f := 1 - e$. Let
$$\hat{\mathbb{A}} = f\hat{\mathbb{A}}f \oplus f\hat{\mathbb{A}}e \oplus e\hat{\mathbb{A}}e \oplus e\hat{\mathbb{A}}f = \mathbb{A}_{00} \oplus \mathbb{A}_{01} \oplus \mathbb{A}_{11} \oplus \mathbb{A}_{10}$$
with $\mathbb{A}_{ij} = \{x \in \hat{\mathbb{A}} \mid ex = ix, xe = jx\}$ the associated Peirce decomposition. Then
$$(\mathbb{A}^+, \mathbb{A}^-) := (\mathbb{A}_{01}, \mathbb{A}_{10}), \quad < xyz >^+ := xyz, \quad < xyz >^- := zyx$$
is an associative pair.

Algebra imbeddings of associative pairs. It is not difficult to show that every associative pair arises from an associative algebra $\hat{\mathbb{A}}$ with idempotent e in the way just described (see [Lo75], Notes to Chapter II). We call $(\hat{\mathbb{A}}, e)$ an *algebra imbedding for* $(\mathbb{A}^+, \mathbb{A}^-)$. There are several such imbeddings (see [Ca04] for a comparison of some of them). The in a certain sense minimal choice for $(\hat{\mathbb{A}}, e)$ is called the *standard imbedding of the associative pair.* For instance, in Example (2) an algebra imbedding is obtained by taking $\hat{\mathbb{A}} = \mathrm{End}(E \oplus F)$ with e the projector onto E along F (but this choice will in general not be minimal). In Example (1), take $\hat{\mathbb{A}} := \mathrm{End}_{\mathbb{A}}(\mathbb{A} \oplus \mathbb{A}) = M(2,2;\mathbb{A})$ and e the projector onto the first factor.

The associated Jordan pair. Formally, associative pairs give rise to Jordan pairs in exactly the same way as grouds give rise to symmetric spaces: the Jordan pair is $(V^+, V^-) := (\mathbb{A}^+, \mathbb{A}^-)$ with the quadratic map $Q^\pm(x)y := < xyx >^\pm$ and its polarized version

$$T^\pm(x,y,z) := Q^\pm(x+z)y - Q^\pm(x)y - Q^\pm(z)y = < xyz >^\pm + < zyx >^\pm .$$

Associative pairs with invertible elements. We call $x \in \mathbb{A}^\pm$ *invertible* if

$$Q(x) : \mathbb{A}^\mp \to \mathbb{A}^\pm, \quad y \mapsto < xyx >$$

is an invertible operator. As shown in [Lo75], associative pairs with invertible elements correspond to unital associative algebras: namely, x is invertible if and only if the algebra $\mathbb{A}_x$ has a unit (which is then $x^{-1} := Q(x)^{-1}x$).

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