An Artinian theory for Lie algebras

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ABSTRACT: Complemented Lie algebras are introduced in this paper (a notion similar to that studied by O. Loos and E. Neher in Jordan pairs). We prove that a Lie algebra is complemented if and only if it is a direct sum of simple nondegenerate Artinian Lie algebras. Moreover, we classify simple nondegenerate Artinian Lie algebras over a field of characteristic 0 or greater than 7, and describe the Lie inner ideal structure of simple Lie algebras arising from simple associative algebras with nonzero socle.

Introduction

The module-theoretic characterization of semiprime Artinian rings (R is unital and completely reducible as a left R-module) cannot be translated to Jordan systems by merely replacing left ideals by inner ideals: if we take, for instance, the Jordan algebra $M_2(F)^{(+)}$ of 2×2-matrices over a field F, any nontrivial inner ideal of $M_2(F)^{(+)}$ has dimension 1, so it cannot be complemented as a F-subspace by any other inner ideal. Nevertheless, O. Loos and E. Neher succeeded in getting the appropriate characterization by introducing the notion of kernel of an inner ideal [**15**]:

A Jordan pair $V = (V^+, V^-)$ (over an arbitrary ring of scalars) is a direct sum of simple Artinian nondegenerate Jordan pairs if and only if it is complemented in the following sense: for any inner ideal B of V^σ there exists an inner ideal C

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of $V^{-\sigma}$ such that each of them is complemented as a submodule by the kernel of the other. In particular, a simple Jordan pair is complemented if and only if is nondegenerate and Artinian.

Since there exists a suitable notion of inner ideal for Lie algebras [**4**], it seems natural to think that the direct sum of simple nondegenerate Artinian Lie algebras can be characterized in a way similar to that of the Jordan pairs. In fact, the half of the work required to prove this conjecture was already done in [**9**], where we showed that any abelian inner ideal of finite length of a nondegenerate Lie algebra L (over a ring of scalars Φ in which $2, 3, 5$ are invertible) is the right wing of a short grading of L.

Let M be an inner ideal of a Lie algebra L. The *kernel* of M

$$\operatorname{Ker} M = \{x \in L : [M, [M, x]] = 0\}$$

is a Φ-submodule of L. A *partner* of M is an inner ideal N of L such that

$$L = M \oplus \operatorname{Ker} N = N \oplus \operatorname{Ker} M.$$

A Lie algebra L is said to be *(weakly) complemented* if any (abelian) inner ideal M of L has a (abelian) partner. Our main result (Theorem 3.7), which can be regarded as a Lie equivalent of the module-theoretic characterization of semiprime Artinian rings, proves the equivalence of the following conditions:

(i) *L is complemented.*

(ii) *L is a direct sum of simple nondegenerate Artinian Lie algebras.*

Moreover, complemented Lie algebras are weakly complemented.

A key tool used in the proof of this result is the notion of subquotient of a Lie algebra with respect to an abelian inner ideal. These subquotients are Jordan pairs which, on the one hand, inherit regularity conditions from the Lie algebra, and, on the other hand, keep the inner ideal structure of L within them. This fact turns out to be crucial for using results of the Jordan theory. For instance, it is used in Theorem 3.5 to prove that any prime weakly complemented Lie algebra satisfies the ascending and descending chain conditions on abelian inner ideals. While any nondegenerate Artinian Jordan pair is a direct sum of finitely many simple nondegenerate Artinian Jordan pairs, a nondegenerate Artinian Lie algebra only has essential socle [**5**, Corollary 3.7]. In fact, there exist strongly prime finite dimensional Lie algebras (over a field of characteristic $p > 5$) with nontrivial ideals [**18**, p. 152]. Therefore, unlike the Jordan case, nondegenerate Artinian Lie algebras are not necessarily complemented: they are only weakly complemented.

In Section 4 we study the inner ideal structure of Lie algebras of traceless operators of finite rank which are continuous with respect to an infinite dimensional pair of dual vector spaces over a division algebra, and of Lie algebras of finite rank skew operators on an infinite dimensional self dual vector space (extending the work of G. Benkart [**3**] for the finite dimensional case, and a previous one of the authors [**8**] for finitary Lie algebras). These results are used in the proof of Theorem 5.3 which describes the simple nondegenerate Artinian Lie algebras over a field of characteristic 0 or greater than 7. Summarizing, we can say that, as conjectured by G. Benkart in the introduction of [**4**], inner ideals in Lie algebras play a role analogous to Jordan inner ideals in the development of an Artinian theory for Lie algebras.

1. **Lie algebras and Jordan pairs**

1.1 *Basic notions.* Throughout this paper, and at least otherwise specified, we will be dealing with Lie algebras L [**12**; **17**] (with $[x, y]$ denoting the Lie bracket and ad_x the adjoint map determined by x), associative algebras R (with product denoted by juxtaposition, xy), Jordan algebras J (with Jordan product written by $x \bullet y$), and Jordan pairs $V = (V^+, V^-)$ [**13**] (with Jordan triple products $\{x, y, z\}$, for $x, z \in V^\sigma$, $y \in V^{-\sigma}$, $\sigma = \pm$) over a ring of scalars Φ containing $\frac{1}{30}$.

1.2 *Nondegeneracy and primeness.* Let $V = (V^+, V^-)$ be a Jordan pair. An element $x \in V^\sigma$, $\sigma = \pm$, is called an *absolute zero divisor* if $Q_x = 0$, and V is said to be *nondegenerate* if it has no nonzero absolute zero divisors, *semiprime* if $Q_{B^\pm} B^\mp = 0$ implies $B = 0$, and *prime* if $Q_{B^\pm} C^\mp = 0$ implies $B = 0$ or $C = 0$, for any ideals $B = (B^+, B^-)$, $C = (C^+, C^-)$ of V. Similarly, given a Lie algebra L, $x \in L$ is an *absolute zero divisor* if $\mathrm{ad}_x^2 = 0$, L is *nondegenerate* if it has no nonzero absolute zero divisors, *semiprime* if $[I, I] = 0$ implies $I = 0$, and *prime* if $[I, J] = 0$ implies $I = 0$ or $J = 0$, for any ideals I, J of L. A Jordan pair or Lie algebra is *strongly prime* if it is prime and nondegenerate. A Lie algebra is *simple* if it is nonabelian and contains no proper ideals.

1.3 Ideals of nondegenerate (strongly prime) Jordan pairs inherit nondegeneracy (strong primeness) [**13**, JP3; **16**]. The same is true for Lie algebras: every ideal of a nondegenerate (strongly prime) Lie algebra is nondegenerate (strongly prime) [**21**, Lemma 4; **10**, 0.4, 1.5].

1.4 [**11**, Theorem 1.6]. A Lie algebra L (over an arbitrary ring of scalars) is strongly prime if and only if $[x,[y,L]]=0$ implies $x=0$ or $y=0$, for every $x,y\in L$.

1.5 *Inner ideals and Jordan elements.* Given a Jordan pair $V=(V^+,V^-)$, an *inner ideal* of V is any Φ-submodule B of V^σ such that $\{B,V^{-\sigma},B\}\subset B$. Similarly, an *inner ideal* of a Lie algebra L is a Φ-submodule B of L such that $[B,[B,L]]\subset B$. An *abelian inner ideal* is an inner ideal B which is also an abelian subalgebra, i.e., $[B,B]=0$.

1.6 There is a natural connection between Lie algebras and Jordan pairs via the notion of abelian inner ideal. Any pair (B,C) of abelian inner ideals of a Lie algebra L becomes a Jordan pair under the triple products defined by

$$\begin{aligned}\{b_1,c,b_2\}:&=[[b_1,c],b_2] \quad \text{for all } b_1,b_2\in B \text{ and } c\in C\\ \{c_1,b,c_2\}:&=[[c_1,b],c_2] \quad \text{for all } c_1,c_2\in C \text{ and } b\in B.\end{aligned}$$

1.7 $x\in L$ is called a *Jordan element* if $\operatorname{ad}_x^3=0$. By [**4**, 1.7(iii)], any Jordan element $x\in L$ satisfies the following analogue of the Jordan identity:

$$\operatorname{ad}^2_{\operatorname{ad}^2_x y}=\operatorname{ad}^2_x\operatorname{ad}^2_y\operatorname{ad}^2_x$$

for any $y\in L$. Clearly, any element of an abelian inner ideal is a Jordan element. Conversely, [**4**, 1.8], any Jordan element yields the abelian inner ideals $[b]:=[b,[b,L]]$ and $(b):=\Phi b+[b]$.

1.8 *Annihilators.* Given a subset S of L, the *annihilator* or *centralizer* of S in L, $\operatorname{Ann}_L S$, consists of the elements $x\in L$ such that $[x,S]=0$. By the Jacobi identity, $\operatorname{Ann}_L S$ is a subalgebra of L and an ideal whenever S is so. Clearly, $\operatorname{Ann}_L L=Z(L)$, the center of L.

1.9 Let I be an ideal of a nondegenerate Lie algebra L. Then, by [**7**, 2.5]

(i) $\operatorname{Ann}_L I=\{a\in L\mid [a,[a,I]]=0\}$. Moreover,

(ii) $I\cap\operatorname{Ann}_L I=0$.

(iii) The factor Lie algebra $L/\operatorname{Ann}_L I$ is nondegenerate.

(iv) I is essential as an ideal iff $\mathrm{Ann}_L I = 0$ iff I is essential as an inner ideal.

(v) L is prime iff if the annihilator of every nonzero ideal of L is zero.

1.10 *Socle and chain conditions.* (i) Recall that the *socle* of a nondegenerate Jordan pair V is $\mathrm{Soc}\, V = (\mathrm{Soc}\, V^+, \mathrm{Soc}\, V^-)$ where $\mathrm{Soc}\, V^\sigma$ is the sum of all minimal inner ideals of V contained in V^σ [**14**]. The *socle* of a nondegenerate Lie algebra L is $\mathrm{Soc}\, L$, defined as the sum of all minimal inner ideals of L [**5**].

(ii) By [**14**, Theorem 2] (for the Jordan pair case) and [**5**, Theorem 3.6] (for the Lie case), the socle of a nondegenerate Jordan pair or Lie algebra is the direct sum of its simple ideals. Moreover, each simple component of $\mathrm{Soc}\, L$ is either inner simple or contains an abelian minimal inner ideal [**4**, Theorem 1.12].

(iii) A Lie algebra L or Jordan pair V is said to be *Artinian* if it satisfies the descending chain condition on all inner ideals.

(iv) A properly ascending chain $0 \subset M_1 \subset M_2 \subset \cdots \subset M_n$ of inner ideals of a Lie algebra L has length n. The *length* of an inner ideal M is the supremum of the lengths of chains of inner ideals of L contained in M.

1.11 *Idempotents and von Neumann regular elements.* (i) Following [**5**], a pair of elements (e, f) of L is said to be an *idempotent* if they satisfy:

$$\mathrm{ad}_e^3 = \mathrm{ad}_f^3 = 0,\ [[e, f], e] = 2e \text{ and } [[e, f], f] = -2f.$$

Notice that the last two conditions imply that $(e, [e, f], f)$ is a $\mathfrak{sl}(2)$-*triple*.

(ii) The term idempotent was borrowed from the terminology of Jordan pairs (cf. [**13**, 5.1]): an *idempotent* of a Jordan pair V is a pair $(e^+, e^-) \in V^+ \times V^-$ such that $Q_{e^+} e^- = e^+$ and $Q_{e^-} e^+ = e^-$.

1.12 Recall (cf. [**5**]) that an element $x \in L$ is called *von Neumann regular* if x is a Jordan element and satisfies $x \in \mathrm{ad}_x^2 L$.

(i) Any von Neumann element e of L can be extended to an idempotent (e, f). In fact (see [**19**, V.8.2] or [**5**, Proposition 2.3]), for each $h \in [e, L]$ such that $[h, e] = 2e$, there exists $f \in L$ satisfying $[e, f] = h$, $\mathrm{ad}_f^3 = 0$ and $[h, f] = -2f$.

(ii) Any idempotent (e, f) yields a 5-grading $L = L^h_{-2} \oplus L^h_{-1} \oplus L^h_0 \oplus L^h_1 \oplus L^h_2$, where each L^h_i is the eigenvalue of the ad-semisimple element $h := [e, f]$ relative to the eigenvalue i, $i = 0, \pm 1, \pm 2$. Moreover, $L^h_2 = [e]$ and $L^h_{-2} = [f]$.

1.13 *Kernels and subquotients.* Let $V = (V^+, V^-)$ be a linear Jordan pair and $B \subset V^+$ an inner ideal of V. Following [**15**], the *kernel* of B is the set

$\operatorname{Ker}_V B = \{x \in V^- : Q_B x = 0\}$. Then $(0, \operatorname{Ker}_V B)$ is an ideal of the Jordan pair (B, V^-) and the quotient $S = (B, V^-)/(0, \operatorname{Ker}_V B) = (B, V^-/\operatorname{Ker}_V B)$ is called the *subquotient of V with respect to B*. The kernel and the corresponding subquotient of an inner ideal $B \subset V^-$ are defined similarly.

There is no problem in extending the notion of kernel to inner ideals of Lie algebras, replacing the Jordan triple product $\{x, y, z\}$ by the left double commutator $[[x, y], z]$. However, to yield a good subquotient, i.e., one which is a Jordan pair, we must restrict ourselves to *abelian* inner ideals.

1.14 Let M be an inner ideal of a Lie algebra L. The *kernel* of M is the set

$$\operatorname{Ker}_L M = \{x \in L : [M, [M, x]] = 0\}.$$

If M is abelian, then we have by [**9**, Lemma 2.3]:

(i) $\operatorname{Ker}_L M = \{x \in L : [m, [m, x]] = 0 \text{ for every } m \in M\}$, and

(ii) $[M, L] + [[L, M], \operatorname{Ker}_L M] + [[\operatorname{Ker}_L M, M], L] \subset \operatorname{Ker}_L M$.

1.15 [**9**, Proposition 2.5]. For any abelian inner ideal M of L, the pair of Φ-modules $V = (M, L/\operatorname{Ker}_L M)$ with the triple products given by

$$\begin{aligned} \{m, \overline{a}, n\} &:= [[m, a], n] \quad \text{for every } m, n \in M \text{ and } a \in L \\ \{\overline{a}, m, \overline{b}\} &:= \overline{[[a, m], b]} \quad \text{for every } m \in M \text{ and } a, b \in L, \end{aligned}$$

where $\overline{x}$ denotes the coset of x relative to the submodule $\operatorname{Ker}_L M$, is a Jordan pair called the subquotient of L with respect to M.

1.16 [**9**, Proposition 2.8]. Let M be an abelian inner ideal of a Lie algebra L, $K = \operatorname{Ker}_L M$ the kernel of M, and $V = (M, L/K)$ the subquotient of L relative to M.

(i) A Φ-submodule B of M is an inner ideal of L if and only if it is an inner ideal of V.

(ii) If C is an inner ideal of L, then $\overline{C} = (C + K)/K$ is an inner ideal of V.

(iii) If L is nondegenerate (strongly prime), then V is nondegenerate (strongly prime).

If L is nondegenerate, then,

(iv) V has nonzero socle if and only if M contains minimal inner ideals. In fact, $\operatorname{Soc} M = \operatorname{Soc} L \cap M$, and

(v) M has finite length if and only if V is Artinian. In this case, $M \subset \operatorname{Soc} L$ and $V \cong (M, I/\operatorname{Ker}_I M)$, where I is any ideal of L containing M.

(vi) If L is strongly prime and M is nonzero and of finite length, then V is a simple nondegenerate Artinian Jordan pair.

2. **Partnered inner ideals**

2.1 Definition. An inner ideal M of a Lie algebra L will be said to be *partnered* if there exists an inner ideal N of L such that

$$L = M \oplus \operatorname{Ker}_L N = N \oplus \operatorname{Ker}_L M.$$

Then N is called a *partner* of M. By symmetry of the definition, N is also partnered with M as a partner.

2.2 Let $V = (V^+, V^-)$ a Jordan pair. According to [**15**], an inner ideal $B \subset V^\sigma$ is said to be *complemented* if there exists an inner ideal $C \subset V^{-\sigma}$ (called a *complement of* B) such that

$$V^\sigma = B \oplus \operatorname{Ker}_V C \text{ and } V^{-\sigma} = C \oplus \operatorname{Ker}_V B.$$

While there is no risk of confusion between the notion of complementation for inner ideals of a Jordan pair and the usual one for Φ-submodules (since B and C do not share the same room), the terms complemented and complement seem us not to be suitable when referred to inner ideals of a Lie algebra (N does not complement M as a Φ-submodule).

2.3 Lemma. *Let $L = L_{-n} \oplus \cdots \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus \cdots \oplus L_n$ be a $(2n+1)$-grading of a Lie algebra L. If L is nondegenerate, then the abelian inner ideals L_n and L_{-n} are partnered by each other*

Proof. By [**9**, 2.6(ii)], $\operatorname{Ker}_L L_n = L_{-(n-1)} \oplus \cdots \oplus L_0 \oplus L_1 \cdots \oplus L_n$, so $L = L_{-n} \oplus \operatorname{Ker}_L L_n$. Similarly, $L = L_n \oplus \operatorname{Ker}_L L_{-n}$. ∎

2.4 Proposition. *Let L be a nondegenerate Lie algebra and I an ideal of L. Then $\operatorname{Ker}_L I = \operatorname{Ann}_L I$. If I is partnered, then $L = I \oplus \operatorname{Ann}_L I$ and I is the only partner of I.*

Proof. Let $x \in \operatorname{Ker}_L I$. Then $[[x, I], I] = 0$ and hence $\operatorname{ad}^2_{[x,I]}(I) = 0$, which implies $[x, I] = 0$ by nondegeneracy of I (1.3). Therefore, $\operatorname{Ker}_L I \subset \operatorname{Ann}_L I$; the reverse inclusion is trivial.

Let C be a partner of I in L, i.e., $L = I \oplus \mathrm{Ker}_L C = C \oplus \mathrm{Ker}_L I$. Since $\mathrm{Ann}_L I$ is an ideal of L, for any $c \in C$ we have

$$\mathrm{ad}_c^2(\mathrm{Ann}_L I) \subset C \cap \mathrm{Ann}_L I = C \cap \mathrm{Ker}_L I = 0.$$

Hence $c \in \mathrm{Ann}_L \mathrm{Ann}_L I$ by 1.9(i). Then, by the modular law, $L = C \oplus \mathrm{Ann}_L I$ implies $C = \mathrm{Ann}_L \mathrm{Ann}_L I$. Then C is an ideal and therefore,

$$\mathrm{Ker}_L C = \mathrm{Ann}_L C = \mathrm{Ann}_L \mathrm{Ann}_L \mathrm{Ann}_L I = \mathrm{Ann}_L I.$$

Thus, $L = I \oplus \mathrm{Ann}_L I$, and $I = \mathrm{Ann}_L \mathrm{Ann}_L I$ is the only partner of I. ■

2.5 Proposition. *Let L be a strongly prime Lie algebra and B an inner ideal of L. If $\mathrm{Ker}_L B \neq 0$, then B is abelian. If B is partnered, then the following conditions are equivalent:*

(i) *B is abelian,*

(ii) *B is proper,*

(iii) *any partner of B is abelian.*

Proof. Note first that for any inner ideal B of a Lie algebra L,

$$[[B,[B,B]],L] \subset [B,B].$$

Indeed, let $b_1, b_2, b_3 \in B$ and $a \in L$. Then

$$\begin{aligned}[[b_1,[b_2,b_3]],a] &= [[b_1,a],[b_2,b_3]] + [b_1,[[b_2,b_3],a]] = [[[b_1,a],b_2],b_3] \\ &\quad + [b_2,[[b_1,a],b_3]] + [b_1,[[b_2,a],b_3]] + [b_1,[b_2,[b_3,a]]] \subset [B,B].\end{aligned}$$

Now if $0 \neq x \in \mathrm{Ker}_L B$, then $[x,[[B,[B,B]],L]] \subset [x,[B,B]] \subset [[x,B],B] = 0$, and hence $[B,[B,B]] = 0$ by (1.4). Now, $[[B,B],[[B,B],L]] \subset [[B,B],B] = 0$, and since L is nondegenerate, $[B,B] = 0$, i.e., B is abelian.

(i) $\Rightarrow$ (ii). Strongly prime (nontrivial) Lie algebras are not abelian.

(ii) $\Rightarrow$ (iii). Let C be a partner of B. If C is not abelian, then $\mathrm{Ker}_L C = 0$ by above. Hence $B = B \oplus \mathrm{Ker}_L C = L$, a contradiction.

(iii) $\Rightarrow$ (i). If B is not abelian, then $\mathrm{Ker}_L B = 0$, and $C = C \oplus \mathrm{Ker}_L B = L$, again a contradiction. ■

2.6 Definition. An abelian inner ideal will be said to be *strongly partnered* if it has an abelian partner.

Any nondegenerate associative or Jordan pair is regular von Neumann if and only if every principal inner ideal is complemented. The same happens for Lie algebras, where the notion of complementation has been replaced by partnership.

2.7 PROPOSITION. *Let L be a nondegenerate Lie algebra and $e \in L$ a Jordan element. Then e is von Neumann regular if and only if the abelian inner ideal $[e] = \mathrm{ad}_e^2 L$ is strongly partnered.*

Proof. If e is von Neumann regular, then it can be extended to an idempotent (e, f) with associated 5-grading $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$, $L_2 = [e]$ and $L_{-2} = [f]$. It follows from (2.3) that $[f]$ is a partner of $[e]$.

Suppose conversely that $[e]$ is partnered by the abelian inner ideal N. Then $L = [e] \oplus \mathrm{Ker}_L N$ and hence there exist $b \in L$ and $x \in \mathrm{Ker}_L N$ such that $e = [e, [e, b]] + x$. By 1.14(ii), $[x, [x, N]] \subset \mathrm{Ker}_L N$ (because N is abelian). Moreover, $x = e - [e, [e, b]] \in (e)$ implies $[x, [x, L]] \subset [e]$. Thus $[x, [x, N]] \subset [e] \cap \mathrm{Ker}_L N = 0$. On the other hand, since $\mathrm{Ker}_L[e] = \mathrm{Ker}_L(e)$ by [**9**, 2.13], $[x, [x, \mathrm{Ker}_L[e]]] = 0$. Therefore, $[x, [x, L]] = [x, [x, N \oplus \mathrm{Ker}_L[e]]] = 0$, which implies $x = 0$ by nondegeneracy of L, so $e = [e, [e, b]]$ is von Neumann regular. ■

2.8 Let M, N be abelian inner ideals of L which are partnered by each other. Then

$$(1_M, \varphi) : (M, L/\mathrm{Ker}_L M) \to (M, N),$$

where φ is the canonical linear isomorphism of $L/\mathrm{Ker}_L M$ onto N, is an isomorphism of Jordan pairs (cf. (1.6) and (1.15)).

3. **Complemented Lie algebras**

3.1 DEFINITION. A Lie algebra L will be called *complemented* if any inner ideal of L has a partner, and L will be called *weakly complemented* if any abelian inner ideal of L has an abelian partner.

At first sight, a complemented Lie algebra seems not to be necessarily weakly complemented. However, as will be seen later, complemented Lie algebras are actually weakly complemented.

3.2 LEMMA. *Let L be a Lie algebra such that any abelian inner ideal has a partner. Then L is nondegenerate.*

Proof. Let $x \in L$ be an absolute zero divisor. Then $M := \Phi x$ is an abelian inner ideal of L with $\mathrm{Ker}_L M = L$. Let N be a partner of M. Then $L = N \oplus$

$\operatorname{Ker}_L M = N \oplus L$ implies $N = 0$, hence $L = M \oplus \operatorname{Ker}_L N$ implies $M = 0$, so $x = 0$ and L is nondegenerate. ∎

3.3 THEOREM. *Suppose that L is complemented and let I be an ideal of L. Then I is a complemented Lie algebra.*

Proof. Let B be an inner ideal of I. Since L is nondegenerate by (3.2), $L = I \oplus \operatorname{Ann}_L(I)$, and B is in fact an inner ideal of L. Let C be a partner of B in L and $\pi_1 : I \oplus \operatorname{Ann}_L(I) \to I$ denote the projection of L onto I. For any inner ideal M of L, $\pi_1(M)$ is an inner ideal of I satisfying the conditions:

$$\operatorname{Ker}_I \pi_1(M) = \operatorname{Ker}_L M \cap I = \pi_1(\operatorname{Ker}_L M).$$

Hence, by the modular law, $L = B \oplus \operatorname{Ker}_L C$ implies $I = B \oplus (\operatorname{Ker}_L C \cap I) = B \oplus \operatorname{Ker}_I \pi_1(C)$, and $L = C \oplus \operatorname{Ker}_L B$ implies $I = \pi_1(L) = \pi_1(C) \oplus \pi_1(\operatorname{Ker}_L B) = \pi_1(C) \oplus \operatorname{Ker}_I B$. Therefore, $\pi_1(C)$ is a partner of B in I. ∎

3.4 PROPOSITION. *Suppose that L is weakly complemented. For any abelian inner ideal M of L, the subquotient $V = (M, L/\operatorname{Ker}_L M)$ is a complemented Jordan pair.*

Proof. Let B be an inner ideal of V contained in M. Then B is an abelian inner ideal of L and therefore it has an abelian partner, say C, in L. Let $\pi : L \to L/\operatorname{Ker}_L M$ denote the canonical projection. We claim that $\pi(C) = (C + \operatorname{Ker}_L C)/\operatorname{Ker}_L C$ is a complement of B in V. By (1.16)(ii), $\pi(C)$ is an inner ideal of V. Moreover, for any $m \in M$, $m \in \operatorname{Ker}_V \pi(C)$ if and only if

$$[[C, m], C] \subset \operatorname{Ker}_L M \cap C \subset \operatorname{Ker}_L B \cap C = 0.$$

Therefore, $\operatorname{Ker}_V \pi(C) = \operatorname{Ker}_L C \cap M$. Hence, by the modular law,

$$M = M \cap (B \oplus \operatorname{Ker}_L C) = B \oplus (M \cap \operatorname{Ker}_L C) = B \oplus \operatorname{Ker}_V \pi(C).$$

On the other hand, $L = C \oplus \operatorname{Ker}_L B$ implies $L/\operatorname{Ker}_L M = \pi(C) + \pi(\operatorname{Ker}_L B) \subset \pi(C) + \operatorname{Ker}_V B$, but this sum is direct by [**15**, Lemma 3.1] since V is nondegenerate.

Via the isomorphism $(M, L/\operatorname{Ker}_L M) \cong (L/\operatorname{Ker}_L N, N)$, cf. (2.8), one proves as before that any inner ideal of V contained in $L/\operatorname{Ker}_L M$ has also a complement. ∎

3.5 THEOREM. *Any prime weakly complemented Lie algebra L satisfies the ascending and descending chain conditions on abelian inner ideals.*

Proof. Let $\{M_i\}$ be a descending or ascending chain of abelian inner ideals of L. Then $M = \bigcup M_i$ is an abelian inner ideal determining the subquotient $V = (M, L/\operatorname{Ker}_L M)$. But V is a complemented Jordan pair by (3.4), and hence V coincides with its socle by [**15**, 5.9]. Since V is also strongly prime by (3.2) and (1.16)(iii), it is necessarily simple. Then, by [**15**, 5.2], V satisfies both chain conditions on inner ideals. Since every M_i is an inner ideal of V, the chain $\{M_i\}$ becomes stationary. ■

3.6 LEMMA. *Let $L = \oplus M_\alpha$ be a direct sum of ideals each of which is a simple nondegenerate Artinian Lie algebra. Then any inner ideal B of L is of the form $B = \oplus B_\alpha$, where for each index α, either $B_\alpha = M_\alpha$ or B_α is an abelian inner ideal of M_α.*

Proof. For each index α, denote by π_α the projection of $L = \oplus M_\alpha$ onto M_α. Then $B_\alpha := \pi_\alpha(B)$ is an inner ideal of M_α. If $B_\alpha = M_\alpha$, then $M_\alpha = [M_\alpha, [M_\alpha, M_\alpha]] = [B, [B, M_\alpha]] \subset B$. Suppose then that B_α is a proper inner ideal of M_α. Then B_α is abelian by [**4**, Lemma 1.13]. Moreover, M_α is simple Artinian, so the subquotient $(B_\alpha, M_\alpha/\operatorname{Ker}_{M_\alpha} B_\alpha)$ coincides with its socle, and any element $b_\alpha \in B_\alpha$ is von Neumann regular in M_α [**14**, Theorem 1]: for each $b_\alpha \in B_\alpha$ there exists $x \in M_\alpha$ such that $b_\alpha = [b_\alpha, [x, b_\alpha]] \in [B, [M_\alpha, B]] \subset B$, which completes the proof. ■

3.7 THEOREM. *For a Lie algebra L, the following notions are equivalent:*

(i) *L is complemented.*

(ii) *L is a direct sum of ideals each of which is a simple nondegenerate Artinian Lie algebra.*

Moreover,

(iii) *Complemented Lie algebras are weakly complemented.*

Proof. *(i) ⇒ (ii).* Assume that L is complemented. It follows from (2.4) that the lattice $\mathcal{I}(L)$ of its ideals is a Boolean algebra, and from standard techniques (see [**15**, 5.1]) that $\mathcal{I}(L)$ is also atomic, so $L = \bigoplus M_\alpha$, where each M_α is a simple complemented Lie algebra (3.3). But M_α is weakly complemented by (2.5), and hence Artinian and nondegenerate by (3.5) and (3.2).

(ii) ⇒ (i). Consider first the case that L is simple, and let B be an inner ideal of L. If $B = L$ then B is self-partnered, so we may assume that B is proper. Then B is an abelian by [**4**, Lemma 1.13], and has finite length since L is Artinian.

Hence, by [**9**, Theorem 3.8], there exists a short grading $L = L_{-n} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_n$ such that $B = L_n$, which implies that B is partnered by (2.3). Consider now the general $L = \bigoplus M_\alpha$, a direct sum of ideals each which is a simple nondegenerate Artinian Lie algebra, and let B be an inner ideal of L. By (3.6), $B = \bigoplus B_\alpha$, where for each index α, either $B_\alpha = M_\alpha$ or B_α is an abelian inner ideal of M_α. By the simple case we have previously analyzed, each B_α has a partner C_α in M_α. Then $C := \bigoplus C_\alpha$ is a partner of B in L, and L is complemented.

(iii) Note that if B is actually abelian, then each B_α is abelian and hence so is C_α by (2.5), which proves that complemented Lie algebras are weakly complemented. ■

3.8 Remarks. While any nondegenerate Artinian Jordan pair is a direct sum of finitely many simple Artinian Jordan pairs, and hence it is complemented and coincides with its socle, a nondegenerate Artinian Lie algebra only has essential socle [**5**, Corollary 3.7]. In fact, there exist strongly prime finite dimensions Lie algebras (over a field of characteristic $p > 5$) with nontrivial ideals [**18**, p. 152]. Therefore, unlike the Jordan case, nondegenerate Artinian Lie algebras are not necessarily complemented.

Nevertheless, nondegenerate Artinian Lie algebras L are actually weakly complemented, as discussed in $(ii) \Rightarrow (i)$ above: if B is an abelian inner ideal of L, then B has an abelian partner by [**9**, Theorem 3.8] and (2.3), since it has finite length.

4. Inner ideal structure of Lie algebras of finite rank operators

We examine in this section the inner ideal structure of simple nondegenerate Lie algebras L arising from simple associative algebras R with nonzero socle. This extends the case where R is Artinian (due to Benkart [**3**]), and that where L is a finitary Lie algebra over a field of characteristic zero (due to the authors [**8**]), equivalently, R satisfies a generalized polynomial identity and so the division associative algebra uniquely determined by R is finite dimensional over its center [**2**, Theorem 6.1.6].

4.1 *Inner ideals of Lie algebras of traceless finite rank operators.* Let $\mathcal{P} = (X, Y, g)$ be a pair of dual vector spaces over a division algebra Δ. A linear operator $a : X \to X$ is *continuous* relative to (X, Y, g) if there exists $a^\# : Y \to Y$ such that $g(ax, y) = g(x, a^\# y)$, for all $x \in X$, $y \in Y$. We denote by $\mathcal{L}_Y(X)$ the (associative) algebra of all continuous operators $a : X \to X$, and by $\mathcal{F}_Y(X)$ the ideal of those continuous operators having finite rank.

By [**2**, Theorem 4.3.8], R is a simple associative algebra coinciding with its socle if and only if it is isomorphic to some $\mathcal{F}_Y(X)$. Moreover, $\mathcal{L}_Y(X)$ can be regarded as the Martindale symmetric ring quotients of $\mathcal{F}_Y(X)$.

(i) For $x \in X$, $y \in Y$, write y^*x to denote the linear operator defined by $y^*x(x') = g(x', y)x$ for all $x' \in X$. Then $y^*x \in \mathcal{F}_Y(X)$.

(ii) Given the subspaces V, W of X, Y respectively, we write W^*V to denote the linear span of the operators w^*v, for all $v \in V$, $w \in W$.

4.2 PROPOSITION. *Let Δ be a division associative algebra over a field of characteristic not 2 or 3, and let $\mathcal{P} = (X, Y, g)$ be an infinite dimensional pair of dual vector spaces over Δ. A subspace B of $[\mathcal{F}_Y(X), \mathcal{F}_Y(X)]$ is a proper inner ideal if and only if $B = W^*V$, where V, W are orthogonal.*

Proof. By [**8**, 2.5(i)], W^*V is a proper inner ideal of L. Conversely, set $R := \mathcal{F}_Y(X)$ and $L := [\mathcal{F}_Y(X), \mathcal{F}_Y(X)]$ and consider a proper inner ideal B of L. Since X is infinite dimensional over Δ, $Z(R) = 0$. Moreover, by [**3**, Lemma 3.13 and 3.14], $[B, B] = 0$ and $b^2 = 0$ for any $b \in B$. Then, for any $b, c \in B$ and $a \in R$, we have $[[b, a], c] = bac + cab \in B$, which implies that B is an inner ideal of the of the Jordan algebra $R^{(+)}$ (with Jordan product $x \bullet y = \frac{1}{2}(xy + yx)$). But inner ideals of $R^{(+)}$ are of the form W^*V, for some subspace W of Y and some subspace V of X [**6**, Theorem 3]. Finally, since $b^2 = 0$ for any $b \in B$, $g(V, W) = 0$. ∎

4.3 Let (X, Y, g) be a pair of dual vector spaces over Δ and set $R = \mathcal{F}_Y(X)$. For $e, f \in \mathcal{L}_Y(X)$, $eRf = (f^{\#}Y)^*(eX)$. Moreover, the subspaces $V = eX$ and $W = f^{\#}Y$ are orthogonal if and only if $fe = 0$. Hence the following statements are equivalent:

(i) R is Artinian,

(ii) R has no infinite sequence of nonzero orthogonal idempotents,

(iii) (X, Y, g) has no pairs of subspaces (V, W), where $V \leq X$ and $W \leq Y$ are infinite dimensional and orthogonal, $g(V, W) = 0$.

4.4 COROLLARY. *Let L be a Lie algebra of the form $[R, R]/Z(R) \cap [R, R]$, where R is a simple associative algebra coinciding with its socle. Then L is Artinian if and only if R is Artinian.*

Proof. If R is Artinian, then L is Artinian by [**3**, Corollary 5.2]. The reverse implication follows from (4.3). ■

4.5 *Inner ideals of Lie algebras of finite rank skew-symmetric operators.* Recall [**2**, Theorem 4.6.8] that a simple associative algebra R with an involution $*$ has nonzero socle if and only if it is $*$-isomorphic to the algebra of finite rank continuous operators $(\mathcal{F}_X(X), *)$, where X is a left vector space endowed with a nondegenerate Hermitian or skew-Hermitian form h ($h(x, y) = \epsilon\overline{h(y, x)}$, $\epsilon = \pm 1$) over a division algebra with involution $(\Delta, -)$. Moreover, the involution $*$ is the adjoint involution with respect to h. Actually, we may assume, without loss of generality, that either h is symmetric (in this case Δ is a field with the identity as involution), or h is skew-Hermitian [**8**, 3.4].

4.6 Let X be a left vector space endowed with a nondegenerate Hermitian or skew-Hermitian form h over a division algebra with involution $(\Delta, -)$. For $x, y \in X$,

(i) $x^* y \in \mathcal{F}_X(X)$ with $(x^* y)^* = \epsilon y^* x$ ($\epsilon = \pm 1$), and

(ii) $[x, y] := x^* y - \epsilon y^* x \in \mathrm{Skew}(\mathcal{F}_X(X), *)$.

Given V, W subspaces of X, we write

(iii) $V^* W$ to denote the linear span of the operators $v^* w$, for all $v \in V$, $w \in W$,

(iv) $[V, W]$ to denote the linear span of the skew-traces $[v_i, w_i]$, $v_i \in V$, $w_i \in W$.

4.7 Proposition. *Let $(\Delta, -)$ be a division associative algebra with involution over a field of characteristic not 2 or 3, X an infinite dimensional left vector space endowed with a nondegenerate symmetric or skew-Hermitian form h over $(\Delta, -)$, and $L = [\mathrm{Skew}(\mathcal{F}_X(X), *), \mathrm{Skew}(\mathcal{F}_X(X), *)]$. Then a subspace B of L is a proper inner ideal of L if and only if B is of one of the following types:*

(i) *$B = [V, V]$ where V is a totally isotropic subspace of X,*

(ii) *Δ is a field with the identity as involution, h is symmetric, and $B = [x, H^\perp]$, where H is hyperbolic plane and x is a nonzero isotropic vector of H.*

Moreover, the length of $[x, H^\perp]$ is 1 if and only if $H^\perp$ is anisotropic; otherwise $[x, H^\perp]$ has length 2.

Proof. By [**8**, 3.6(i) and 3.7(i)], both $[V, V]$ and $[x, H^\perp]$ are proper inner ideals of L. Conversely, set $R := \mathcal{F}_X(X)$ and $K := \mathrm{Skew}(\mathcal{F}_X(X), *)$.

We will use a direct limit argument to reduce the question to the case when

X is finite dimensional. Let X_α be a finite dimensional subspace of X such that $\dim_{Z(\Delta)} X_\alpha > 4$, and the restriction h_α of h to $X_\alpha \times X_\alpha$ is nondegenerate. Then $X = X_\alpha \oplus X_\alpha^\perp$. Set $R_\alpha := \mathcal{L}_{X_\alpha}(X_\alpha)$ and denote by L_α the set of all $a \in L$ such that $aX_\alpha^\perp = 0$. Then L_α is a subalgebra of L isomorphic to $[K_\alpha, K_\alpha]$, $K_\alpha = \mathrm{Skew}(R_\alpha, *)$. Since B is proper, we can take a directed set $\{R_\alpha\}$ such R is the direct limit of the R_α and no L_α is contained in B. Then each $B_\alpha := B \cap L_\alpha$ is a proper inner ideal of $L_\alpha \cong [K_\alpha, K_\alpha]$. By [**3**, 4.21 and 4.26], B_α is abelian, and by [**3**, 4.23], $b^3 = 0$ for any $b \in B_\alpha$. Since $Z(R_\alpha)$ is a field, $B_\alpha \cap Z(R_\alpha) = 0$ and we can regard B_α as an abelian inner ideal of the *simple* Lie algebra $[K_\alpha, K_\alpha]/[K_\alpha, K_\alpha] \cap Z(R_\alpha)$, via the isomorphism: $B_\alpha \cong (B_\alpha + Z(R_\alpha))/Z(R_\alpha)$. Now we consider two cases.

(*Case 1*) $b^2 = 0$ for any $b \in B_\alpha$ and for all indexes α. Then it follows from the proof of [**3**, Theorem 5.5] that $B_\alpha = e_\alpha R_\alpha e_\alpha^*$, where e_α is an idempotent of R_α such that $e_\alpha^* e_\alpha = 0$. It is easy to see that $V_\alpha := e_\alpha X_\alpha$ is a totally isotropic vector subspace. Moreover, by [**8**, 3.6(ii)], $B_\alpha = e_\alpha R_\alpha e_\alpha^* = [V_\alpha, V_\alpha]$. Since the R_α form a directed set, so do the V_α. Thus, $V := \cup V_\alpha$ is a totally isotropic subspace of X, and $B = [V, V]$.

(*Case 2*) $b^2 \neq 0$ for some $b \in B_\alpha$ and some index α. Then it follows from the proof of [**3**, Theorem 5.5] that Δ is a field B with the identity as involution. Hence, we have by [**8**, 3.8(ii)] (where only $\mathrm{char}(F) \neq 2$ is used in the proof) that $B = [x, H^\perp]$ as in (ii). Moreover, by [**8**, 3.7(iv)], $B = [x, H^\perp]$ is minimal, equivalently, has length 1, if and only if $H^\perp$ is anisotropic; otherwise B has length 2 (any inner ideal of $L = \mathfrak{so}(X, h)$ strictly contained in $[x, H^\perp]$ is of the form $\Delta[x, z]$ for some isotropic vector $z \in H^\perp$). ■

4.8 A right ideal I of an associative algebra with involution $(R, *)$ will be called *isotropic* if $I^* I = 0$. Then $(R, *)$ will be called isotropic if contains nonzero isotropic right ideals.

4.9 Lemma. *Let X be a left vector space endowed with a nondegenerate Hermitian or skew-Hermitian form h over a division algebra with involution $(\Delta, -)$, and let $(R, *)$ be the associative algebra $\mathcal{F}_X(X)$ with the adjoint involution. Then a right ideal I of R is isotropic if and only if $I = X^* V$ for some totally isotropic subspace V of X.*

Proof. Right ideals of $\mathcal{F}_X(X)$ are of the form $I = X^* V$, where V is a subspace of X. Now $I^* I = (V^* X)(X^* V) = X^* h(V, V) X = 0$ if and only if V is totally isotropic. ■

4.10 Corollary. *Let $(\Delta, -)$ be an associative algebra with involution over a field of characteristic not 2 or 3, X an infinite dimensional left vector space endowed with a nondegenerate symmetric or skew-Hermitian form h over $(\Delta, -)$, $R = \mathcal{F}_X(X)$, and $L = [\mathrm{Skew}(\mathcal{F}_X(X), *), \mathrm{Skew}(\mathcal{F}_X(X), *)]$, with $*$ being the adjoint involution. Then the following conditions are equivalent:*

(i) *(X, h) does not contain infinite dimensional totally isotropic subspaces.*

(ii) *$(R, *)$ satisfies the descending chain condition on isotropic right ideals.*

(iii) *L is Artinian.*

5. **Simple nondegenerate Artinian Lie algebras**

We give in this section a structure theorem for simple nondegenerate Artinian Lie algebras over a field of characteristic 0 or greater than 7.

5.1 A Lie algebra L will be called a *division Lie algebra* if it is nonzero, nondegenerate and has no nontrivial inner ideals.

5.2 *Examples of division Lie algebras.*

(i) Let Δ be a division associative algebra such that $[[\Delta, \Delta], \Delta] \neq 0$. Then $[\Delta, \Delta]/[\Delta, \Delta] \cap Z(\Delta)$ is a division Lie algebra, [**3**, Corollary 3.15].

(ii) Let R be a simple associative algebra with involution $*$ and nonzero socle. Suppose that $Z(R) = 0$ or the dimension of R over $Z(R)$ is greater than 16, and set $K := \mathrm{Skew}(R, *)$. Then $L = [K, K]/[K, K] \cap Z(R)$ is a division Lie algebra if and only if $(R, *)$ has no nonzero isotropic right ideals. This is a direct consequence of the inner ideal structure of L: [**3**, Theorem 5.5] when R is Artinian, and (4.7) when R is not Artinian.

5.3 Theorem. *Let L be a simple Lie algebra over a field F of characteristic 0 or greater than 7. Then L is Artinian and nondegenerate if and only if it is one of the following:*

(i) *A division Lie algebra.*

(ii) *A (finite dimensional over its centroid) simple exceptional Lie algebra.*

(iii) *$[R, R]/[R, R] \cap Z(R)$, where R is a simple Artinian associative algebra.*

(iv) *$[K, K]/[K, K] \cap Z(R)$, where $K = \mathrm{Skew}(R, *)$ and R is a simple associative algebra with involution $*$ which coincides with its socle, such that $Z(R) = 0$*

or the dimension of R over $Z(R)$ is greater than 16*, and* $(R, *)$ *satisfies the descending chain condition on isotropic right ideals.*

Proof. Let us first see that any of the Lie algebras listed above has the required properties: (i) this is clear for division Lie algebras; (ii) any simple exceptional Lie algebra L is finite dimensional over its centroid C, so it is Artinian, and for the algebraic closure $\overline{C}$ of C, $\overline{C} \otimes L$ is nondegenerate (see, for instance, [**18**, Theorem 3]); (iii) $[R, R]/[R, R] \cap Z(R)$ is nondegenerate by [**5**, 5.2], and Artinian by (4.4); (iv) $[K, K]/[K, K] \cap Z(R)$ is nondegenerate by [**5**, 5.9], and Artinian by [**3**, Corollary 5.6] if R is Artinian, and by (4.10) if R is not Artinian.

Suppose, conversely, that L is a simple nondegenerate Artinian Lie algebra. Then L contains minimal inner ideal and by [**5**, 6.3] it is one of the following: (i) a division Lie algebra; (ii) a simple exceptional Lie algebra; (iii) $[R, R]/[R, R] \cap Z(R)$, with R a simple associative algebra with nonzero socle; (iv) $[K, K]/[K, K] \cap Z(R)$, where $K = \mathrm{Skew}(R, *)$ and R is a simple associative algebra such that $Z(R) = 0$ or the dimension of R over $Z(R)$ is greater than 16, has an involution $*$, and coincides with its socle. Therefore, we only need to deal with the cases (iii) and (iv). By (4.4), if $[R, R]/[R, R] \cap Z(R)$ is Artinian then R is Artinian. Finally, it follows from (4.10) that if $[K, K]/[K, K] \cap Z(R)$ is Artinian, then $(R, *)$ satisfies the descending chain condition on isotropic right ideals. ■

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