Irreducible bimodules over alternative algebras and superalgebras

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Abstract

The irreducible alternative superbimodules are studied. The complete classification is obtained for even bimodules of arbitrary dimension and for finite-dimensional irreducible superbimodules over an algebraically closed field.

1   An Introduction

Alternative algebras are the nearest generalization of associative algebras. An algebra A is called alternative if it satisfies the identities

$$\begin{aligned} (x, x, y) &= 0 \text{ (left alternativity)}, \\ (x, y, y) &= 0 \text{ (right alternativity)}, \end{aligned}$$

where $(x, y, z) = (xy)z - x(yz)$ is the associator of the elements x, y, z. A classical example of nonassociative alternative algebra is the 8-dimensional algebra of octonions (Cayley numbers), or, more generally, a Cayley-Dickson algebra.

The structure theory of finite-dimensional alternative algebras was developed by M. Zorn: if A is a finite-dimensional alternative algebra, then A contains a unique maximal nilpotent ideal R (the nil radical of the algebra A), and the quotient algebra A/R is semisimple [29]. Every semisimple algebra is a direct sum of ideals which are simple algebras [28]. Every simple alternative algebra is either an associative algebra or a Cayley-Dickson algebra over its

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center (see [13, 28]). For principal results of the theory of infinite-dimensional alternative algebras we refer the reader to [27].

The theory of representations of alternative algebras was initiated in the papers by R.D.Schafer [14] and N. Jacobson [6], where btrepresentations of finite-dimensional alternative algebras were studied. In particular, in [14] it has been shown that any birepresentation of a semisimple alternative algebra of characteristic 0 (without any restriction on dimension) is completely reducible, and in [6] the irreducible alternative bimodules over finite-dimensional algebras were classified. It should be noted that, contrary to the associative case, for nonassociative algebras the concept of birepresentation is more natural and easily defined than that of one-sided (right or left) representation. Right representations of alternative algebras were studied in K. A. Zhevlakov [26] and A. M. Slin'ko, I. P. Shestakov [20]. In particular, in [20] some description of right irreducible alternative modules was obtained and a characterization of the quasi-regular radical in terms of representation theory was given. However, even in the finite-dimensional case the classification of irreducible alternative right modules remains incomplete. For example, until now it is not clear whether the left regular representation of a Cayley-Dickson algebra (it is irreducible) is an alternative right representation.

The simple alternative superalgebras were described by E. I. Zel'manov, I. P. Shestakov [25] (in the case of characteristic different from 2 and 3) and I. P. Shestakov [17] (for arbitrary characteristic). It turned out that nontrivial examples of such superalgebras appear in the case of characteristic 2 or 3 only. The study of representations of alternative superalgebras began in N. A. Pisarenko [9]. In that paper the finite-dimensional irreducible alternative superbimodules over finite-dimensional simple alternative superalgebras of characteristic different from 2 and 3 were described. It turned out that in this case nontrivial alternative non-associative superbimodules appear in the case of the minimal nontrivial simple superalgebra, i.e. doubled field $F[u] = F \oplus Fu$, $u^2 = 1$ only. In [7, 21] irreducible bimodules over the simple alternative superalgebras $B(1|2)$ and $B(4|2)$ of characteristic 3 and of dimensions 3 and 6 were described.

The aim of our paper is to describe irreducible birepresentations of alternative algebras of arbitrary dimension and characteristic, on one side, and on the other, to describe finite-dimensional irreducible alternative superbimodules. We also describe irreducible superbimodules of any dimension and characteristic over the simple alternative superalgebras. To complete the description of irreducible infinite-dimensional superbimodules, it remains to describe such

superbimodules (or to prove their absence) over prime locally nilpotent alternative superalgebras of characteristic 3.

The results of the paper were announced in [22].

2 Definitions and known results

2.1 Birepresentations of alternative algebras

Let A be an alternative algebra over a field F and V be an A-bimodule, i. e. V is a vector space over F such that on V left and right bilinear multiplications by the elements of the algebra A are defined:

$$A \otimes_F V \to V, \quad (a \otimes v) \mapsto a \cdot v, \quad V \otimes_F A \to V, \quad (v \otimes a) \mapsto v \cdot a,$$

where $a \in A$, $v \in V$. In the vector space direct sum $E = A \oplus V$ define a product $*$ by

$$(a_1 + v_1) * (a_2 + v_2) = a_1 a_2 + a_1 \cdot v_2 + v_1 \cdot a_2,$$

where $a_1, a_2 \in A$, $v_1, v_2 \in V$. So E becomes an algebra over F, in which A is a subalgebra and V is an ideal with zero multiplication. The algebra E is called the *split null extension* of the algebra A by its bimodule V. The bimodule V is called an *alternative A-bimodule*, if the split null extension $E = A \oplus V$ is an alternative algebra. It is easy to check that a bimodule V is alternative if and only if for any $a, b \in A$, $v \in V$ the following equalities holds in the algebra E

$$(a, b, v) = (b, v, a) = (v, a, b), \quad (v, a, a) = 0.$$

The description of irreducible (bi)modules is one of the main problem in the (bi)representation theory in any class of algebras. For an algebra A denote by $\operatorname{Reg} A$ the *regular bimodule* $V = A$ with the action of A given by the product in A. It is clear that the bimodule $\operatorname{Reg} \mathbb{O}$ over a Cayley-Dickson algebra $\mathbb{O}$ is an irreducible faithful alternative bimodule which is not associative. Another examples of alternative nonassociative bimodules provide the so-called Cayley bimodules over a generalized quaternion algebra $\mathbb{H}$, which are defined as follows. Let $a \mapsto \bar{a}$ be the symplectic involution in $\mathbb{H}$, then an alternative $\mathbb{H}$-bimodule V is called a *Cayley bimodule* if it satisfies the identity $av = v\bar{a}$ for any $a \in \mathbb{H}$, $v \in V$. The Cayley $\mathbb{H}$-bimodules form a category which is isomorphic to the category of left associative $\mathbb{H}$-modules via the following isomorphism (see [17, Lemmas 11,12]). Let L be a left associative $\mathbb{H}$-module

with the action $(a, v) \mapsto av$. Then the vector space L with the bimodule operations

$$v \cdot a = av, \quad a \cdot v = \bar{a}v,$$

is an alternative Cayley bimodule over $\mathbb{H}$ which we will denote as Cay L.

The results of the mentioned in the introduction papers of R. D. Schafer [14] and N. Jacobson [6] lead to the following description of finite-dimensional irreducible alternative bimodules.

Theorem 2.1. *Let A be a finite-dimensional alternative algebra over a field F of characteristic different from 2 and V be an irreducible alternative faithful A-bimodule. Then one of the following cases holds*:

- *A is associative and V is an associative bimodule,*

- *$A = \mathbb{O}$ and $V = \text{Reg}\,\mathbb{O}$,*

- *$A = \mathbb{H}$ and $V = \text{Cay}\,L$, where L is a minimal left ideal of $\mathbb{H}$.*

Irreducible bimodules over generalized quaternion algebras were described in [6] under the assumption that the ground field has characteristic different from 2 and centralizes the bimodule. In [17] the second author extended this result, removing the restrictions on the characteristic, dimension, and the condition of centralization.

2.2 Alternative superalgebras

An algebra A is called a $\mathbb{Z}_2$-graded algebra or *superalgebra*, if $A = A_{\bar{0}} \oplus A_{\bar{1}}$, where $A_i A_j \subseteq A_{i+j}$, $i, j \in \mathbb{Z}_2$. The subspaces $A_{\bar{0}}$ and $A_{\bar{1}}$ are called respectively the even and odd parts of the superalgebra A. For example, the Grassmann algebra $G = G_{\bar{0}} \oplus G_{\bar{1}}$ becomes a superalgebra, if we denote by $G_{\bar{0}}$ ($G_{\bar{1}}$) the submodule generated by the words of even (odd) length from generators of the algebra G. A superalgebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$ is called an *alternative superalgebra*, if its Grassmann envelope $G(A) = G_{\bar{0}} \otimes A_{\bar{0}} + G_{\bar{1}} \otimes A_{\bar{1}}$ is an alternative algebra. From this definition it follows that the alternative superalgebras are defined by the identities

$$\begin{aligned}
(a_i, a_j, a_k) &= (-1)^{jk+1}(a_i, a_k, a_j), \\
(a_i, a_j, a_k) &= (-1)^{ij+1}(a_j, a_i, a_k), \\
(a_0, a_0, A) &= 0,
\end{aligned}$$

where $a_s \in A_s$, $s = i, j, k \in \{\bar{0}, \bar{1}\}$.

Let A be an algebra. Denote by $A[u]$ the superalgebra obtained by doubling of the algebra A:

$$A[u] = A \oplus A \cdot u, \quad (A[u])_{\bar{0}} = A, \quad (A[u])_{\bar{1}} = A \cdot u,$$

where u is an odd central element, $u^2 = \alpha \in F$, $\alpha \neq 0$. When $u^2 = 1$, we will denote $A[u]$ as $A[\sqrt{1}]$.

C. T. C. Wall [23] proved that every simple finite-dimensional associative superalgebra over an algebraically closed field F is isomorphic to one of the following superalgebras:

- $A = \mathrm{M}_{m|n}(F), \quad A_{\bar{0}} = \left\{ \begin{pmatrix} \star & 0 \\ 0 & \star \end{pmatrix} \begin{matrix} m \\ n \end{matrix} \right\}, \quad A_{\bar{1}} = \left\{ \begin{pmatrix} 0 & \star \\ \star & 0 \end{pmatrix} \begin{matrix} m \\ n \end{matrix} \right\},$

- $A = \mathrm{M}_n(F)[\sqrt{1}],$ the doubled matrix algebra.

In [25] it was proved that every nontrivial simple alternative superalgebra of characteristic different from 2 and 3 is associative. In particular, the nontrivial simple finite-dimensional alternative superalgebras over an algebraically closed field F of characteristic different from 2, 3 are exhausted by the superalgebras $\mathrm{M}_{m|n}$, $\mathrm{M}_n[\sqrt{1}]$.

The simple alternative superalgebras of characteristic 2 and 3 were described in [17]. In these cases nonassociative nontrivial simple superalgebras appeared.

Over a field of characteristic 2, any such a superalgebra is isomorphic to one of the following two superalgebras obtained from a Cayley-Dickson algebra:

- the Cayley-Dickson superalgebra $\mathbb{O}(4|4) = \mathbb{O} = \mathbb{H} + v\mathbb{H}$ with the $\mathbb{Z}_2$-grading induced by the Cayley-Dickson process applying to the generalized quaternion subalgebra $\mathbb{H}$,

- the doubled Cayley-Dickson algebra $\mathbb{O}[u]$.

Note that both these superalgebras are alternative algebras.

A simple nonassociative nontrivial alternative superalgebra of characteristic 3 is isomorphic to one of the following superalgebras (see [16, 17]):

- *superalgebra* $B(1|2)$. Let V be a two-dimensional vector space over a field F with a nonzero skew-symmetric bilinear form $\langle \cdot, \cdot \rangle$. The unital superalgebra $B(1|2) = F \cdot 1 \oplus V$ with an identity 1 is given by the grading

$$B(1|2)_{\bar{0}} = F \cdot 1, \quad B(1|2)_{\bar{1}} = V$$

and the supercommutative multiplication $xy = \langle x, y \rangle \cdot 1$ for $x, y \in V$;

- *superalgebra* $B(4|2)$. Let V be the same space as above; set

$$B(4|2)_{\bar{0}} = \operatorname{End} V \cong \operatorname{M}_2(F), \quad B(4|2)_{\bar{1}} = V$$

  with the multiplication

$$v \cdot a = a(v) = \bar{a} \cdot v, \quad u \cdot v = \langle \cdot, u \rangle v \in \operatorname{End} V,$$

  where $a \in \operatorname{End} V$, $u, v \in V$ and $a \mapsto \bar{a}$ denote the symplectic involution in $\operatorname{End} V$ (i. e. $\langle a(u), v \rangle = \langle u, \bar{a}(v) \rangle$ for any $u, v \in V$). It is easy to check that $V = B(4|2)_{\bar{1}}$ is a Cayley bimodule over $\operatorname{M}_2(F) = B(4|2)_{\bar{0}}$;

- *twisted superalgebra of vector type* $B(\Gamma, D, \gamma)$. Let Γ be an associative commutative algebra over a field F, D be a nonzero derivation of Γ, $\gamma \in \Gamma$. Denote by $\bar{\Gamma}$ an isomorphic copy of the vector space Γ with respect to the isomorphism $a \mapsto \bar{a}$ and set $B(\Gamma, D, \gamma) = \Gamma \oplus \bar{\Gamma}$ with the multiplication

$$\begin{aligned}
a \cdot b &= ab, \\
a \cdot \bar{b} &= \bar{a} \cdot b = \overline{ij}, \\
\bar{a} \cdot \bar{b} &= \gamma ab + 2D(a)b + aD(b),
\end{aligned}$$

  where $a, b \in \Gamma$, ab is the product of the elements a and b in Γ, and with the grading

$$B(\Gamma, D, \gamma)_{\bar{0}} = \Gamma, \quad B(\Gamma, D, \gamma)_{\bar{1}} = \bar{\Gamma}.$$

  The superalgebra $B(\Gamma, D, \gamma)$ is simple if and only if the algebra Γ does not contain proper D-invariant ideals (i.e. D-simple).

2.3 Alternative superbimodules

The alternative superbimodules are defined similarly to the non-graded case. A bimodule $V = V_{\bar{0}} \oplus V_{\bar{1}}$ over an alternative superalgebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$ is called an *alternative A-superbimodule*, if the split null extension $A \oplus V = (A_{\bar{0}} \oplus V_{\bar{0}}) \oplus (A_{\bar{1}} \oplus V_{\bar{1}})$ is an alternative superalgebra.

We will call an A-superbimodule $V^{\mathrm{op}} = V_{\bar{0}}^{\mathrm{op}} + V_{\bar{1}}^{\mathrm{op}}$ *opposite* to an A-superbimodule $V = V_{\bar{0}} + V_{\bar{1}}$, if $V_{\bar{0}}^{\mathrm{op}} = V_{\bar{1}}$, $V_{\bar{1}}^{\mathrm{op}} = V_{\bar{0}}$ and A acts on it by the following rule: $a \cdot v = (-1)^{|a|} av$, $v \cdot a = va$, where $v \in V^{\mathrm{op}}$, $a \in A_{\bar{0}} \cup A_{\bar{1}}$, and $|a|$ means the parity of a, that is, $|a| = i$ if $a \in A_i$, $i = \bar{0}, \bar{1}$.

It is easy to check that for any superbimodule V the identical application $V \to V^{op}$, $v \mapsto v$, is an odd isomorphism between V and V^{op}. In particular,

if V is alternative, the opposite superbimodule V^{op} is alternative as well. We sometimes will say that the bimodule V^{op} is obtained from V by *changing of parity*.

As in the case of algebras, the main problem of the theory of representations of superalgebras is the description of irreducible superbimodules. For alternative algebras, such bimodules are always defined either over simple algebras or over a direct sum of two simple algebras, and in the latter case the bimodule is associative. It is an important difference of the supercase that there are irreducible superbimodules over nilpotent superalgebras. This first was noted in [16], where this fact was used to construct finite-dimensional solvable non-nilpotent alternative and Jordan superalgebras.

Let $A = A_{\bar{1}} = Fx$, $x^2 = 0$. Assume that the field F contains a primitive 3-th root of 1, i. e. such element ε, that $\varepsilon^2 + \varepsilon + 1 = 0$. Consider the A-bimodule $V^\varepsilon(1|1) = Fv_0 \oplus Fv_1$, where $V_{\bar{0}} = Fv_0$, $V_{\bar{1}} = Fv_1$ and the action of the element u is given by the equalities

$$v_0 \cdot x = v_1, \quad x \cdot v_0 = \varepsilon v_1, \quad v_1 \cdot x = v_0, \quad x \cdot v_1 = (\varepsilon + 1)v_0.$$

It is easy to check that $V^\varepsilon(1|1)$ is an irreducible alternative superbimodule; furthermore, $V^\varepsilon(1|1)^{\mathrm{op}} \cong V^{\varepsilon^2}(1|1)$.

Note that the commutator bimodule $V^\varepsilon(1|1)^{(-)}$ over the Malcev superalgebra $A^{(-)}$ appeared in the paper by A. Elduque and the second author [5].

It is clear that the bimodule $V^\varepsilon(1|1)$ can be considered also as a unital superbimodule over the superalgebra $A^\sharp = F \oplus Fx$ obtained from A by adjoining an identity 1; obviously, it remains irreducible.

If the field F does not contain a primitive 3-th root of 1 and char $F \neq 3$, then a four-dimensional irreducible alternative superbimodule over A can be constructed (see [16]).

It is clear that for any simple alternative superalgebra A the regular superbimodule Reg A and its opposite companion $(\mathrm{Reg}\,A)^{\mathrm{op}}$ are irreducible alternative superbimodules.

N. A. Pisarenko [9] investigated the structure of alternative superbimodules over finite-dimensional semisimple superalgebras of characteristic different from 2, 3.

We call V *nontrivial*, if $V_{\bar{0}} \neq 0$ and $V_{\bar{1}} \neq 0$.

Theorem 2.2 ([9]). *Let A be a finite-dimensional semisimple alternative superalgebra over an algebraically closed field F of characteristic different from 2, 3, thus $A = A_1 \oplus A_2 \oplus \ldots \oplus A_k$, where each summand A_i is isomorphic to either a*

Cayley-Dickson algebra $\mathbb{O}$ or a matrix superalgebra of types $M_{m|n}$ and $M_k[\sqrt{1}]$ over F. Then

- *any alternative superbimodule over A is a direct sum of associative (A_i, A_j)-bimodules and unital alternative A_i-bimodules;*

- *any nontrivial faithful irreducible alternative superbimodule V over the simple superalgebra A_i is either associative and isomorphic to one of the bimodules $\operatorname{Reg} M_{m|n}$, $(\operatorname{Reg} M_{m|n})^{\mathrm{op}}$, $\operatorname{Reg}(M_k[\sqrt{1}])$, $\operatorname{Reg}(M_k[\sqrt{1}])^{\mathrm{op}}$, or $A_i = F[\sqrt{1}] = F[u]$, $\dim V = 2$, $V_{\bar{0}} = Fv_0$, $V_{\bar{1}} = Fv_1$ and the (unital) action of A_i on V has one of the following forms:*

  1. $v_0 u = v_1$, $v_1 u = 0$, $uv_0 = -2v_1$, $uv_1 = -v_0$,

  2. $v_0 u = v_1$, $v_1 u = 2v_0$, $uv_0 = 0$, $uv_1 = v_0$,

  3. $v_0 u = \frac{1+\alpha}{\delta} v_1$, $v_1 u = \delta v_0$, $uv_0 = v_1$, $uv_1 = (1 - \alpha)v_0$,

  where $\alpha, \delta \in F$ are non-zero roots of the equation $\alpha^2 + \alpha\delta + \delta^2 = 1$.

Remark 2.1. *Note that in the theorem above, the bimodules of types 1) - 3) can be defined in a unified manner as the following series of unital bimodules over $F[\sqrt{1}] = F \oplus Fu$:*

$$V = V_{\lambda,\mu}(1|1) = Fv_0 \oplus Fv_1, \quad V_{\bar{0}} = Fv_0, \quad V_{\bar{1}} = Fv_1,$$

with the action

$$v_0 u = (3\mu - \lambda)v_1, \quad uv_0 = 2\lambda v_1, \quad v_1 u = 2\mu v_0, \quad uv_1 = (\lambda + \mu)v_0,$$

where $\lambda, \mu \in F$, $\lambda^2 + 3\mu^2 = 1$. In fact, the bimodule of type 1) is the bimodule $V_{-1,0}(1|1)$, the bimodule of type 2) is isomorphic to the bimodule $V_{0,1/\sqrt{3}}(1|1)$, the bimodules of type 3) are isomorphic to $V_{\lambda,\mu}(1|1)$ for $\lambda = \sqrt{2 - 2\alpha - \delta}/2$, $\mu = \delta/(4\lambda)$.

The irreducible unital superbimodules over the superalgebras $B(1|2)$ and $B(4|2)$ were described by M. C. López-Díaz and the second author in [7]. (The classification of irreducible finite-dimensional alternative superbimodules over $B(1|2)$ was also obtained independently by the first author in [21].)

Theorem 2.3 ([7, 21]). *Any irreducible alternative bimodule over the superalgebra $B = B(1|2)$, which is not associative, is isomorphic either to $\operatorname{Reg} B$, or to $(\operatorname{Reg} B)^{\mathrm{op}}$, or belongs to the series of irreducible bimodules*

$$V(\lambda, \mu)(3|3) = V_{\bar{0}} \oplus V_{\bar{1}},$$
$$V_{\bar{0}} = \operatorname{Vect}_F \langle v_0, v_1 R_y, v_0 R_y^2 \rangle, \quad V_{\bar{1}} = \operatorname{Vect}_F \langle v_1, v_0 R_y, v_1 R_y^2 \rangle,$$

with the action

$$vR_y^j \cdot y = \begin{cases} vR_y^{j+1}, & j < 2, \\ \mu v^s, & j = 2, \end{cases}$$

$$vR_y^j \cdot x = \lambda v^s R_y^j + j v R_y^{j-1}, \quad j = 0, 1, 2,$$

where $v \in \{v_0, v_1\}$, $v_i^s = v_{1-i}$ and λ, μ are nonzero scalars. In addition, the superbimodules $V(\lambda, \mu)$ and $V(\lambda', \mu')$ are isomorphic if and only if $(\lambda, \mu) = \pm(\lambda', \mu')$. Moreover, $V(\lambda, \mu) \cong \big(V(\lambda, \mu)\big)^{\mathrm{op}}$.

Theorem 2.4 ([7]). *Every alternative bimodule over the superalgebra $B(4|2)$ is completely reducible, and every irreducible bimodule up to the change of parity is isomorphic to the regular bimodule.*

2.4 Useful identities

We recall some useful identities that hold in any alternative algebra (see [3], [17], [27]):

$$[ab, c] = a[b, c] + [a, c]b + 3(a, b, c), \tag{1}$$
$$(ab, c, d) = a(b, c, d) + (a, c, d)b - (a, b, [c, d]), \tag{2}$$
$$[(a, b, c), d] = (ab, c, d) + (bc, a, d) + (ca, b, d), \tag{3}$$
$$((c, a, b), a, b) = [a, b](c, a, b), \tag{4}$$
$$((a, b, c), x, y) = ((a, x, y), b, c) + (a, (b, x, y), c) + (a, b, (c, x, y)) \tag{5}$$
$$\qquad\qquad - [b, (a, c, [x, y])] + ([a, c], b, [x, y]). \tag{6}$$
$$(z, x, ty) = -(z, t, xy) + (z, x, y)t + (z, t, y)x, \tag{7}$$
$$(z, x, yt) = -(z, t, yx) + x(z, t, y) + t(z, x, y).. \tag{8}$$

We will also need the identities

$$z(yxy) = ((zy)x)y, \tag{9}$$
$$(zy)(xz) = (z(yx))z, \tag{10}$$

known as *right* and *central Moufang identities* (see [27]).

3 Reduction to prime superalgebras and superalgebras with $A_1^2 = 0$.

Let A be an alternative superalgebra and V be an irreducible A-superbimodule. Denote by $\mathrm{Ann}\, V$ the annihilator of V: $\mathrm{Ann}\, V = \{a \in A \,|\, a \cdot V = V \cdot a =$

$0\}$. It is easy to see that $\operatorname{Ann} V$ is an ideal of A and V is an irreducible faithful bimodule over $A/\operatorname{Ann} V$. A bimodule V is called *associaive* if $(V, A, A) = 0$. One can easily check that if V is associative than $(A, A, A) \subseteq \operatorname{Ann} V$, hence faithful associative bimodules exist only over associative algebras.

Below V always will denote an irreducible faithful A-superbimodule which is not associative. We will use the following notations:

- $E = E(A, V) = A \oplus V$, the split null extension;

- (λ, ρ), the birepresentation of A associated with V, that is, λ, ρ are linear mappings from A to $\operatorname{End} V$ such that for any $a \in A, \ v \in V$ holds

$$\lambda(a) : v \mapsto (-1)^{|a||v|} a \cdot v, \quad \rho(a) : v \mapsto v \cdot a;$$

- $\mathcal{Z}(V) = \mathcal{Z}_A(V) := \{\varphi \in \operatorname{End} V \mid [\varphi, \rho(a)]_s = [\varphi, \lambda(a)]_s = 0 \ \forall a \in A\}$, the centralizer of the A-bimodule V, where $[a, b]_s = ab - (-1)^{|a||b|} ba$;

- $\mathcal{E} := \mathcal{Z}(\operatorname{Reg} E)$, the supercentroid of E;

- $\Gamma = \{\alpha \in \mathcal{E} \mid V\alpha \subseteq V, \ A\alpha \subseteq A\}$.

Proposition 3.1 ([5]). *The centralizer $\mathcal{Z} = \mathcal{Z}(V)$ is a graded division algebra and Γ is a commutative superalgebra which can be embedded in $\mathcal{Z}$ via the application $\pi : \alpha \mapsto \alpha|_V$.*

Proof. By the graded version of the Schur Lemma, for an irreducible bimodule V the centralizer $\mathcal{Z}(V)$ is a graded division algebra. Consider the homomorphism $\pi : \Gamma \to \operatorname{End} V$, $\alpha \mapsto \alpha|_V$, and let $\alpha \in \ker \pi$. Then $V(A\alpha) = (V\alpha)A = 0$, hence $A\alpha = 0$ and $\alpha = 0$. It is clear also that $\pi(\Gamma) \subseteq \mathcal{Z}$. Furthermore, let $\alpha, \beta \in \Gamma$, $v \in V, x \in A$, then we have

$$
\begin{aligned}
((v\alpha)\beta)x &= (-1)^{|\beta||x|}(v\alpha)(x\beta) = (-1)^{|\beta||x|+|\alpha|(|x|+|\beta|)}(v(x\beta))\alpha \\
&= (-1)^{|\alpha||x|+|\alpha||\beta|}((v\beta)x)\alpha = (-1)^{|\alpha||\beta|}((v\beta)\alpha)x.
\end{aligned}
$$

Therefore, $(V[\alpha, \beta]_s)A = 0$, which implies that $[\alpha, \beta]_s = 0$ and Γ is a super-commutative subsuperalgebra of $\mathcal{Z}$.

$\square$

Theorem 3.1. *Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be an alternative superalgebra. If there exists an irreducible faithful A-superbimodule V which is not associative then A is prime or $A_{\bar{1}} \neq 0$, $A_{\bar{1}}^2 = 0$.*

We first prove two lemmas.

Lemma 3.1. *Let nonzero elements $a, b \in A_{\bar{0}} \cup A_{\bar{1}}$ satisfy $[a, b]_s = (a, A, b) = 0$. Then the subset (V, a, b) is an A-subbimodule of V. Moreover, if $ab = ba = 0$ and at least one of the elements a, b is even than $(V, a, b) = 0$.*

Proof. In fact, for any $v \in V_{\bar{0}} \cup V_{\bar{1}}$, $r \in A_{\bar{0}} \cup A_{\bar{1}}$ we have by (2) and its superization

$$
\begin{aligned}
(v, a, b)r &= \pm(vr, a, b) \pm v(r, a, b) \pm (v, r, [a, b]_s) = \pm(vr, a, b) \in (V, a, b), \\
r(v, a, b) &= (rv, a, b) \pm (r, a, b)v + (r, v, [a, b]_s) = (rv, a, b) \in (V, a, b),
\end{aligned}
$$

which proves that (V, a, b) is a subbimodule of V. Now, let $ab = ba = 0$ and $a \in A_{\bar{0}}$. If $(V, a, b) \neq 0$ than by irreducibility $(V, a, b) = V$. By the Moufang identities, we have $a(v, a, b) = (v, a, ba) = 0$, $(v, a, b)a = (v, a, ab) = 0$. Therefore, $0 \neq a \in \operatorname{Ann} V$, a contradiction. $\square$

Lemma 3.2. *Let nonzero ideals I, J of A satisfy $IJ = JI = 0$. Then $I + J \subseteq A_{\bar{1}}$. Moreover, if $A = A_{\bar{0}}$ then A is prime.*

Proof. Assume that $I_{\bar{0}} \neq 0$. By the previous lemma, the set $N = (V, I, J)$ is an A-subbimodule of V and $(V, I_{\bar{0}}, J) = (V, I, J_{\bar{0}}) = 0$. If $i \in I_{\bar{1}}$, $j \in J_{\bar{1}}$, $a \in I_{\bar{0}}$, then, by the Moufang identities, for any homogeneous $v \in V$

$$
\begin{aligned}
(v, i, j)a &= (v, i, aj) + (v, a, ij) + (v, a, j)i = 0, \\
a(v, i, j) &= (v, i, ja) - (v, a, ji) - (-1)^{|v|}i(v, a, j) = 0.
\end{aligned}
$$

Thus $0 \neq I_{\bar{0}} \subseteq \operatorname{Ann} N$, and $N = 0$. Consider now the sets $I \cdot V$, $V \cdot I$. It is easy to see that they are subbimodules of V and at least one of them is nonzero. Assume that $I \cdot V \neq 0$, then $I \cdot V = V$ and we have $J \cdot V = J \cdot (I \cdot V) \subseteq (JI) \cdot V + (J, I, V) = 0$. Similarly, if $V \cdot I \neq 0$ then we would have $V \cdot J = 0$ and $0 \neq J \subseteq \operatorname{Ann} V = 0$, a contradiction. Therefore, $V \cdot I = 0$, and consequently $(V, A, I) = 0$. Furthermore, by the Moufang identity we have

$$
I \cdot (V, A, A) \subseteq (V, A, AI) + (V, I, AA) + A \cdot (V, I, A) = 0,
$$

which proves that $(V, A, A) \subseteq \operatorname{Ann}_V(I) = \{v \in V \mid v \cdot I = I \cdot v = 0\}$. It is clear that $\operatorname{Ann}_V(I)$ is a subbimodule of V which should be zero since $I \neq 0$. Therefore, $(V, A, A) = 0$ which contradicts to non-associativity of V. The contradiction proves that $I_{\bar{0}} = 0$. Similarly, $J_{\bar{0}} = 0$. In particular, if I is an ideal of A with $I^2 = 0$ then $I \subseteq A_{\bar{1}}$.

We continue by considering the case $A = A_{\bar{0}}$. Assume that I, J are ideals of A with $IJ = 0$. Then $(I \cap J)^2 = 0$, $I \cap J \subseteq A_{\bar{1}} = 0$, and we have $JI \subseteq I \cap J = 0$. Therefore, $I = 0$ or $J = 0$, and A is prime. $\qquad\square$

The proof of the theorem. Assume that $A_{\bar{1}}^2 \neq 0$ and prove that in this case A is prime. As above, it suffices to prove that if I, J are ideals in A with $IJ = JI = 0$ then $I = 0$ or $J = 0$. Assume that $I \neq 0$, $J \neq 0$, then by Lemma 3.2, we have $I + J \subseteq A_{\bar{1}}$. Therefore, $IA_{\bar{1}} + A_{\bar{1}}I \subseteq I_{\bar{0}} = 0$ and $0 \neq A_{\bar{1}} \subseteq \operatorname{Ann} I$. Since the annihilator of an ideal in A is an ideal, we may apply Lemma 3.2 to the ideals $I, \operatorname{Ann} I$ and to get $\operatorname{Ann} I \subseteq A_{\bar{1}}$. Then $A_{\bar{1}}^2 \subseteq \operatorname{Ann} I \subseteq A_{\bar{1}}$ and $A_{\bar{1}}^2 = 0$, a contradiction. $\qquad\square$

4   The case $A_{\bar{1}} = 0$.

The objective of this section is to generalize the results by N.Jacobson and R.D.Schafer on irreducible alternative bimodules given in Theorem 2.1 to arbitrary dimension and characteristic.

Theorem 4.1. *Let A be an alternative algebra and V be an irreducible alternative faithful A-bimodule which is not associative, both A and V to be of arbitrary dimension. Then one of the following cases holds*:

- $A = \mathbb{O}$ *and* $V = \operatorname{Reg} \mathbb{O}$,

- $A = \mathbb{H}$ *and* $V = \operatorname{Cay} L$, *where L is a minimal left ideal of* $\mathbb{H}$.

Proof. By Theorem 3.1, A is prime. Hence, by [27], A is either an associative prime algebra or a Cayley-Dickson ring or is degenerate, that is, contains nonzero absolute zero divisors. (Recall that an element $a \in A$ is called an *absolute zero divisor (a.z.d.)* if $aAa = 0$).

In the last case, let $0 \neq a \in A$ be an a.z.d. We next show that $a \in \operatorname{Ann} V$. Indeed, let $u = v \cdot a \neq 0$ for some $v \in V$. Then V is generated by u. Let $(\lambda, \rho) : A \to \operatorname{End} V$ be the birepresentation of A associated with the A-bimodule V and $M(A)$ be the subalgebra of the associative algebra $\operatorname{End} V$ generated by the set $\{\lambda(a), \rho(a) \,|\, a \in A\}$. Then $V = uM(A)$, and there exists $W \in M(A)$ such that $v = uW = v\rho(a)W = \cdots = v(\rho(a)W)^n$. From [15, Corollary of Lemma 1, Corollary 1 of Theorem 1], it follows that the operator $\rho(a)W$ is nilpotent. Hence $v = 0$, a contradiction.

Therefore, A is a prime associative algebra or a Cayley-Dickson ring. Let us prove that in the first case A is a central order in a generalized quaternion algebra. Show first that A is not commutative. In fact, if it were true then

by Lemma 3.1 for any $a, b \in A$ we would have an A-subbimodule (V, a, b). Since $(V, A, A) \neq 0$, there exist $a, b \in A$ such that $V = (V, a, b)$. But then $V = (V, a, b) = ((V, a, b), a, b) = [a, b](V, a, b) = 0$ by (4), a contradiction.

Let $V_{as} = \{v \in V \,|\, (v, A, A) = 0\}$. Since A is associative, it follows from (2) that V_{as} is a subbimodule of V which should be zero since V is irreducible and not associative. Since A is prime and noncommutative, A does not satisfy the identity $[x, y]^4 = 0$. Let $a, b \in A$ such that $n = [a, b]^4 \neq 0$. Assume that there exists $c \in A$ such that $[n, c] \neq 0$. By [27, Lemma 7.5], we have $((V[n, c], A, A), A, A) = 0$. Therefore, $((V[n, c], A, A) \subseteq V_{as} = 0$ and $V[n, c] \subseteq V_{as} = 0$. Similarly, $[n, c]V = 0$ and $[n, c] \subseteq \operatorname{Ann} V = 0$. The contradiction shows that A satisfies the identity $[[x, y]^4, z] = 0$. Since A is prime, the Posner theorem [12, x] implies that A is a central order in a generalized quaternion algebra.

Therefore, $Z(A) \neq 0$ and the algebra of quotients $(Z^*)^{-1}A$ is a generalized quaternion algebra $\mathbb{H}$ or a Cayley-Dickson algebra $\mathbb{O}$ over $(Z^*)^{-1}Z$. Let us show that V is a $(Z^*)^{-1}A-$bimodule.

Lemma 4.1. *Let A be a prime superalgebra, $Z = Z(A)_{\bar{0}} \neq 0$. Then $\lambda(z) = \rho(z) \in \mathcal{Z}(V)$ for any $z \in Z$ and V is an irreducible faithful $(Z^*)^{-1}A$-bimodule.*

Proof. By Lemma 3.1, for any $z \in Z$, $a \in A$ the set (V, a, z) is an A-subbimodule of V. If $(V, a, z) = V$, then, by (4) we have $V = (V, a, z) = ((V, a, z), a, z) = 0$, a contradiction. Hence $(V, a, z) = 0$, and we have

$$(Vz)a = V(za) = V(az) = (Va)z \subseteq Vz,$$

$$a(Vz) = (aV)z \subseteq Vz,$$

which proves that Vz is an A-subbimodule of V. Since V is faithful, we have $Vz = V$ for any nonzero $z \in Z$. Moreover, $\operatorname{Ann} z$ is an A-subbimodule of V too. Hence $\operatorname{Ann} z = 0$. In other words, the aplication $\rho(z) \in \operatorname{End} V$ is inversible.

We proceed to show that $[V, z] = 0$. By (1), the set $[V, z]$ is an A-subbimodule of V. If $[V, z] = V$, then, by (2), for any $a, b \in A$

$$(V, a, b) = ([V, z], a, b) \subseteq (ab, V, z) + a(b, V, z) + (a, V, z)b = 0,$$

a contradiction. Thus $[V, z] = 0$, $\rho(z) = \lambda(z) \in \mathcal{Z}(V)$ and V has a structure of a vector space over the field $(Z^*)^{-1}Z$. Clearly, V has also a natural structure of a $(Z^*)^{-1}A-$bimodule, which is evidently faithful and irreducible.

$\square$

If $(Z^*)^{-1}A = \mathbb{O}$, a Cayley-Dickson algebra, then $V \cong \operatorname{Reg} \mathbb{O}$, the regular bimodule [6]. Consider an element $m \in V$ corresponding to the identity 1 of $\mathbb{O}$; then $V = m \cdot \mathbb{O}$. On the other hand, $m \cdot A$ is a nonzero A-subbimodule of V. Hence $m \cdot A = V = m \cdot \mathbb{O}$. Consequently, for every $z \in Z$ there exists $a \in A$ such that $m \cdot z^{-1} = m \cdot a$. We have $m \cdot (1 - az) = 0$. Therefore $1 = az$ and $z^{-1} = a$. Thus $A = \mathbb{O}$ is a Cayley-Dickson algebra.

If $(Z^*)^{-1}A = \mathbb{H}$, a generalized quaternion algebra, then V is the Cayley bimodule $\operatorname{Cay} L$, that is, V is isomorphic to the left irreducible associative $\mathbb{H}$-module L on which $\mathbb{H}$ acts as follows: $v \cdot a = av$ and $a \cdot v = \bar{a}v$, where av is the associative action of $\mathbb{H}$ on L and $a \mapsto \bar{a}$ is the symplectic involution in $\mathbb{H}$. In this case, consider $I = \operatorname{Ann}_l L = \{a \in A \mid aL = 0\}$. Obviously, I is an ideal of A and $V \cdot I = 0$. Hence $(A, V, I) = 0$. Using the Moufang identities, we obtain that $(V, A, A)I = I(V, A, A) = 0$. Note that, by (2), (V, A, A) is an $A-$subbimodule of V. It follows that $I = 0$, because V is not associative. Thus L is a faithful irreducible left A-module. Therefore, A is a primitive PI-algebra and, since A can be embedded into $\mathbb{H}$, A is a PI-algebra. By Kaplansky's Theorem, it follows that the center of A is a field. Hence $A = (Z^*)^{-1}A = \mathbb{H}$.

<div align="right">□</div>

5 The case $A_{\bar{1}}^2 = 0, A_{\bar{1}} \neq 0$.

5.1 $A = F\,1 \oplus F\,x,\ x^2 = \lambda$.

We will first consider the minimal possible case when $\dim_F A_{\bar{1}} = 1$, $A_{\bar{1}} = Fx$, $x^2 = 0$. The category of alternative bimodules over the superalgebra $A = A_{\bar{1}} = Fx$ is isomorphic to the category of unital alternative bimodules over the superalgebra $A^\sharp = F1 \oplus Fx$, therefore we will study the last one. In fact, in order to include also Pisarenko's result on bimodules over $F[\sqrt{1}]$, we will consider the more general case when $A = F\,1 \oplus F\,x,\ x^2 = \lambda \in F$.

Recall the definition of the universal multiplicative enveloping superalgebra $\mathcal{U}(A)$ of an alternative superalgebra A. Let $\bar{A}$ be an isomorphic copy of the vector space A under the isomorphism $a \mapsto \bar{a}$. Consider the vector space direct sum $A \oplus \bar{A}$ with the $\mathbb{Z}_2$-grading induced by that on A. Then the tensor algebra $T(A \oplus \bar{A})$ has a natural structure of an associative superalgebra. Denote by

I_{alt} the ideal of $T(A \oplus \bar{A})$ generated by the set of elements

$$a \otimes b - ab + (-1)^{|a||b|}(b \otimes a - ba),$$
$$\bar{a} \otimes \bar{b} - \overline{ab} + (-1)^{|a||b|}(\bar{b} \otimes \bar{a} - \overline{ba}),$$
$$ab - a \otimes b - \bar{a} \otimes b + (-1)^{|a||b|}b \otimes \bar{a}, \qquad (11)$$
$$\overline{ab} + \bar{a} \otimes b - (-1)^{|a||b|}(\bar{b} \otimes \bar{a} + b \otimes \bar{a}),$$
$$c \otimes c - c^2,$$

where $a, b \in A_{\bar{0}} \cup A_{\bar{1}}$, $c \in A_{\bar{0}}$; the elements of the last type are needed only in case of char $F = 2$. Since the generators of the ideal I_{alt} are homogeneous, the quotient algebra $\mathcal{U}(A) = T(A \oplus \bar{A})/I_{alt}$ inherits the superalgebra structure from $T(A)$.

Consider a pair of linear mapping $(\mathcal{L}, \mathcal{R}) : A \to \mathcal{U}(A)$ defined as follows: $\mathcal{L} : a \mapsto \mathcal{L}_a := \bar{a} + I_{alt}$, $\mathcal{R} : a \mapsto \mathcal{R}_a := a + I_{alt}$; then for any alternative birepresentation $(\lambda, \rho) : A \to \mathrm{End}\, V$ there exists a unique (super)algebra homomorphism $\varphi : \mathcal{U}(A) \to \mathrm{End}\, V$ such that $\lambda = \varphi \circ \mathcal{L}$, $\rho = \varphi \circ \mathcal{R}$. Conversely, every homomorphism $\varphi : \mathcal{U}(A) \to \mathrm{End}\, V$ defines a structure of an alternative A-bimodule on V. In other words, the categories $\mathrm{Bimod}_{\mathrm{Alt}}\!-\!A$ and $\mathrm{Mod}_{\mathrm{As}}\!-\!\mathcal{U}(A)$ are isomorphic. The (super)algebra $\mathcal{U}(A)$ is called *the universal multiplicative enveloping (super)algebra* of A.

If A has a unit 1 then the quotient algebra $\mathcal{U}_1(A) = \mathcal{U}(A)/(1 - \mathcal{R}_1, 1 - \mathcal{L}_1)$ is called *the universal unital multiplicative enveloping (super)algebra* of A.

Proposition 5.1. *Let $A = F1 + Fx$, $x^2 = \lambda$, $A_{\bar{0}} = F1$, $A_{\bar{1}} = Fx$. Then the universal unital multiplicative enveloping superalgebra $\mathcal{U}_1(A)$ is a free 4-dimensional module over its center which is isomorphic to the polynomial ring $F[t]$. More exactly, it has a basis $\{1, a, b, ab\}$ over $F[t]$, where $a^2 = t$, $b^2 = t - 2\lambda$, $a \circ b = \lambda - t$.*

If char $F = 3$ and $\lambda = 0$ then $\mathcal{U}_1(A)$ has a graded ideal $I = F[t](ab - t - \lambda) + F[t](a - b)$ such that $I^2 = 0$, and in this case $\mathcal{U}_1(A)/I \cong (F[t])[\sqrt{t}]$.

If char $F \neq 3$ or $\lambda \neq 0$ then $\mathcal{U}_1(A)$ is a prime superalgebra whose central closure is a generalized quaternion algebra over the field $F(t)$.

Proof. Denote $a = \mathcal{R}_x$, $b = \mathcal{L}_x$, then $\mathcal{U}_1(A) = \mathrm{alg}\langle 1, a, b\rangle$. It follows from (11) that the elements a, b satisfy the relations

$$a^2 - \lambda = b^2 + \lambda = -a \circ b.$$

Denote $t = a^2$, then t is an even central element of $\mathcal{U}_1(A)$, and one can easily check that $\mathcal{U}_1(A)$ is spanned as an $F[t]$-module by the elements $1, a, b, ab$.

Moreover, a standart use of the Groebner-Shirshov basis method [2] (known also as The Diamond Lemma) shows that $\mathcal{U}_1(A)$ is a free $F[t]$-module and $F[t]$ is a polynomial ring.

If char $F = 3$ and $\lambda = 0$ then one can easily check that I is an ideal in $\mathcal{U}_1(A)$ and $I^2 = 0$. The quotient superalgebra $\bar{\mathcal{U}} = \mathcal{U}_1(A)/I$ has the form $\bar{\mathcal{U}} = F[t] \oplus F[t]\,\bar{a}$, $\bar{a}^2 = t$, where $\bar{\mathcal{U}}_{\bar{0}} = F[t]$, $\bar{\mathcal{U}}_{\bar{1}} = F[t]\,\bar{a}$.

Now let char $F \neq 3$ or $\lambda \neq 0$. Since $\mathcal{U}_1(A)$ is a free module over the center $F[t]$, we may consider the central closure $\tilde{\mathcal{U}} = (F[t])^{-1}\mathcal{U}_1(A)$ which has the same basis $\{1, a, b, ab\}$ over the field $F(t)$. Denote $u = -\frac{1}{t-\lambda}ab$, then $u^2 = u - \frac{t(t-2\lambda)}{(t-\lambda)^2}$ and the subalgebra $F(t)[u]$ has a structure of a composition algebra over $F(t)$ with the involution $u \mapsto \bar{u} = 1 - u$. One can easily check that $\tilde{\mathcal{U}} \cong (F(t)[u], t)$, that is, $\tilde{\mathcal{U}}$ is obtained from $F(t)[u]$ by the Cayley-Dickson process with the parameter t [27, section 2.2]. Hence $\tilde{\mathcal{U}}$ is a generalized quaternion algebra.

$\square$

We can now classify irreducible bimodules over the superalgebra $A = F1 + Fx$, $x^2 = \lambda$.

Theorem 5.1. *Let V be a unital irreducible faithful alternative non-associative bimodule over the superalgebra $A = F1 + Fx$, $x^2 = \lambda$. Then there exists a simple algebraic field extension $K = F(\alpha)$, $\alpha \neq \lambda$, such that V is a K-vector space of $\dim_K V \leq 4$, and up to the changing of parity we have the following possibilities.*

If the polynomial $f(u) = u^2 + (\alpha - \lambda)u + \alpha(\alpha - 2\lambda)$ has a root $\varepsilon \neq 0$ in K, then $V = Kv \oplus Kvx$, $V_{\bar{0}} = Kv$, $V_{\bar{1}} = Kvx$, and the action of A is given as follows:

$V^{\varepsilon}_{\alpha,\lambda}(1|1):$ *if $\alpha \neq 0$ then*

$$v \cdot x = vx, \quad x \cdot v = \tfrac{\varepsilon}{\alpha}\, vx,$$
$$vx \cdot x = \alpha v, \quad x \cdot vx = (\alpha + \varepsilon - \lambda)\, v;$$

$V_{\lambda}(1|1):$ *if $\alpha = 0$ then*

$$v \cdot x = vx, \quad x \cdot v = -2vx,$$
$$vx \cdot x = 0, \quad x \cdot vx = -\lambda\, v.$$

If the polynomial $f(u)$ above is irreducible over K, then $\mathcal{U}(\alpha) := \mathcal{U}_1(A)/(t - \alpha)$ is a graded division algebra over K, and V is isomorphic to the left regular super-module over $\mathcal{U}(\alpha)$. More exactly,

$V_{\alpha,\lambda}(2|2)$: $\dim_K V = 4$, V has a basis $\{v, xv, vx, x(vx)\}$, $V_{\bar{0}} = Kv \oplus Kx(vx)$, $V_{\bar{1}} = Kvx \oplus Kxv$, and the action of A is given as follows:

$$\begin{aligned}
v \cdot x &= vx, & x \cdot v &= xv, \\
vx \cdot x &= \alpha v, & x \cdot vx &= x(vx), \\
xv \cdot x &= x(vx) + (\lambda - \alpha)v, & x \cdot xv &= (2\lambda - \alpha)v, \\
x(vx) \cdot x &= (\alpha - \lambda)vx + \alpha xv, & x \cdot x(vx) &= (2\lambda - \alpha)vx.
\end{aligned}$$

Conversely, for any simple algebraic field extension $K = F(\alpha)$, $\alpha \neq \lambda$, the above defined superbimodules $V_{\alpha,\lambda}^{\varepsilon}(1|1)$ and $V_{\alpha,\lambda}(2|2)$ are irreducible and alternative.

Proof. Notice first that every irreducible alternative bimodule V over a unital (super)algebra is necessary unital. In fact, by Lemma 4.1, $\lambda(1) = \rho(1) \in \mathcal{Z}(V)$, which implies that $W = \{v - v \cdot 1 \,|\, v \in V\}$ is a subbimodule of V. Moreover, $W \neq V$ since $1 \in \operatorname{Ann} W$ and $\operatorname{Ann} V = 0$. Thus $W = 0$, and V is unital.

Therefore, the bimodule V in the theorem is a unital associative irreducible $\mathcal{U}$-module for $\mathcal{U} = \mathcal{U}_1(A)$. Let $\varphi : \mathcal{U} \to \operatorname{End} V$ be the representation of $\mathcal{U}$ corresponding to the $\mathcal{U}$-supermodule V, then $\varphi(t) = \alpha$ lies in the even part of the centralizer of V which is a graded division algebra. Thus we may consider $\varphi(\mathcal{U}) \subseteq \operatorname{End} V$ as a homomorphic image of the superalgebra $\mathcal{U}(\alpha) = \mathcal{U}/(t - \alpha)$ over the field $K = F(\alpha)$ with the same basis $\{1, a, b, ab\}$, where $a^2 = \alpha$, $b^2 = \alpha - 2\lambda$, $a \circ b = \lambda - \alpha$. It is easy to see that if $\alpha = \lambda$ then V is associative. Thus $\alpha - \lambda \neq 0$. Furthermore, since $\varphi(\mathcal{U})$ is finitely generated over F and finite dimensional over $K = F(\alpha)$, it is easy to see that α is algebraic over F (see, for example, [10, Lemma 1.1]).

If $\operatorname{char} F = 3$ and $\lambda = 0$, then by Proposition 5.1 the superalgebra $\mathcal{U}(\alpha)$ has the ideal $I(\alpha)$ with $I(\alpha)^2 = 0$. Since V is irreducible, $\varphi(I) = 0$, thus $\varphi(\mathcal{U}) \cong \mathcal{U}(\alpha)/I(\alpha) = K \oplus K\bar{a}$, $\bar{a}^2 = \alpha$. It is clear that in this case V as a $\varphi(\mathcal{U})$-module is isomorphic to the regular supermodule, which coincides with the bimodule $V_{\alpha,\lambda}^{\varepsilon}(1|1)$ for $\varepsilon = \alpha$ and $\lambda = 0$.

Let $\operatorname{char} F \neq 3$ or $\lambda \neq 0$; then the superalgebra $\mathcal{U}(\alpha)$ is simple and isomorphic to a generalized quaternion superalgebra over K, with $\mathcal{U}(\alpha)_{\bar{0}} = K1 + Kab$, $\mathcal{U}(\alpha)_{\bar{1}} = Ka + Kb$. Therefore, in this case $\varphi(\mathcal{U}) = \mathcal{U}(\alpha)$, and we have the two possibilities, according to the structure of $\mathcal{U}(\alpha)_{\bar{0}}$.

If $\mathcal{U}(\alpha)_{\bar{0}}$ is a field, that is, the polynomial $f(u) = u^2 + (\alpha - \lambda)u + \alpha(\alpha - 2\lambda)$ is irreducible over K, then $V \cong \operatorname{Reg}\mathcal{U}(\alpha)$ as a right $\mathcal{U}(\alpha)$-supermodule, and $V \cong V_{\alpha,\lambda}(2|2)$ as an A-bimodule.

If the polynomial $f(u)$ has a root ε in K then the K-subspace $L = \mathrm{vect}\langle ab - \varepsilon,\ (\lambda - \alpha - \varepsilon)\,a - \alpha\,b\rangle$ is a minimal graded right ideal of $\mathcal{U}(\alpha)$. It produces the irreducible superbimodule $V_{\alpha,\lambda}^{\varepsilon}(1|1)$. Since $\tilde{\mathcal{U}}$ is simple and finite-dimensional, any other irreducible graded right $\mathcal{U}(\alpha)$-module up to the changing of parity is isomorphic to L. $\qquad\square$

Remark 5.1. *Observe that we have the following isomorphisms:*

$$V_{\alpha,\lambda}^{\varepsilon_1}(1|1)^{\mathrm{op}} \cong V_{\alpha,\lambda}^{\varepsilon_2}(1|1), \quad V_\alpha(2|2)^{\mathrm{op}} \cong V_\alpha(2|2),$$

where $f(u) = (u - \varepsilon_1)(u - \varepsilon_2)$.

Remark 5.2. *If the field F is algebraically closed then we have $K = F$; $\lambda \in \{0,\ 1\}$, $f(u)$ reducible and $\dim_F V = 2$. Moreover, if $\lambda = 0$, we may take $\alpha = 1$, and for $\alpha = 0$ we may take $\lambda = 1$. Thus up to the changing of parity, $V = V_1(1|1)$, $V = V_{\alpha,1}^{\varepsilon}(1|1)$, or $V = V_{1,0}^{\varepsilon}(1|1)$, where in the last case $\varepsilon^2 + \varepsilon + 1 = 0$.*

For $\lambda = 1$ observe also that the bimodules $V_1(1|1)$, $V_{2,1}^0(1|1)$ are isomorphic to the bimodules of type 1,2 from Theorem 2.2, and $V_{\alpha,1}^{\varepsilon}(1|1)$ is isomorphic to the bimodule of type 3 for the pair $(\alpha - 1, \varepsilon)$.

Remark 5.3. *In case when $A = A_1 = Fx$, $x^2 = 0$, we have the isomorphism $\mathcal{U}(A) \cong \mathcal{U}_1(A^\sharp)$, hence A has the same irreducible superbimodules as $A^\sharp$ does.*

5.2 $A_1^2 = 0$, the general case.

We will consider now the general case of superalgebras with $A_1^2 = 0$.

Theorem 5.2. *Let V be an irreducible faithful alternative bimodule over the superalgebra A with $A_1^2 = 0$. Then there exist a field extension K of F, a nonzero algebraic element $\alpha \in K$, an F-subspace T and an F-subalgebra S of K such that:*

    i) V is a unital irreducuble faithful alternative bimodule over the superalgebra $B = K \oplus Ku$, $u^2 = 0$, and the structure of V as a B-bimodule is given by Theorem 5.1, with $\rho(u)^2 = \alpha$.

    ii) $S + Tu$ is an F-subsuperalgebra of B which is isomorphic to A.

    iii) $K = \mathrm{alg}_F\langle \alpha T^2\rangle$.

    iv) The action of A on V is inherited by that of B.

Conversely, let K be a field extension of F, and let T, S and α be an F-subspace, F-subalgebra and F-algebraic element of K such that $ST + TS \subseteq T$ and $\mathrm{alg}_F \langle \alpha T^2 \rangle = K$. Consider the superalgebra $B = K \oplus Ku$, $u^2 = 0$, then $A = S + Tu$ is a subsuperalgebra of B and every irreducible faithful alternative birepresentation (λ, ρ) of B with $\rho(u)^2 = \alpha$ is so over A.

Proof. Consider the subsuperalgebra Γ of the centroid $\mathcal{E} = \mathcal{E}(E)$ of the split null extension $E = V \oplus A$: $\Gamma = \{\alpha \in \mathcal{E} \mid V\alpha \subseteq V, A\alpha \subseteq A\}$. By Proposition 5.1, Γ is supercommutative and isomorphic to a subsuperalgebra of the centralizer $\mathcal{Z} = \mathcal{Z}(V)$ which is a division superalgebra. We will identify Γ with its image in $\mathcal{Z}$.

Let $(L, R) : A \to \mathrm{End}\, E$ be the regular birepresentation of A in E, then for any $x, y \in A_{\bar{1}}$ the operator $D = R_x R_y$ is a derivation of the superalgebra E due to (2). Since $D|_A = 0$, it follows that $D \in \Gamma \subseteq \mathcal{Z}$.

If $R_x R_y = 0$ for all $x, y \in A_{\bar{1}}$ then $(V A_{\bar{1}}) A_{\bar{1}} = 0$. But $V A_{\bar{1}}$ is a subbimodule of V, and if $V = V A_{\bar{1}}$ then $V = (V A_{\bar{1}}) A_{\bar{1}} = 0$, a contradiction. Thus $V A_{\bar{1}} = 0$. It follows from (11) that $L_x L_y = R_x R_y = 0$, hence we have $A_{\bar{1}}(A_{\bar{1}} V) = 0$ and $A_{\bar{1}} V = 0$ as well. Thus $A_{\bar{1}} \subseteq \mathrm{Ann}\, V = 0$, a contradiction.

Let $R_x R_y \neq 0$, show that then $R_x^2 \neq 0$. In fact, we have seen that $0 \neq D = R_x R_y \in \mathcal{Z}$. Therefore, $V = VD = VD^2 = V R_x R_y R_x R_y$. But $R_x R_y = R_y R_x$ by (11), hence $V = V R_x^2 R_y^2$ and $R_x^2 \neq 0$. Since R_x^2 is invertible on V, so is R_x.

Assume now that $R_z R_t = 0$ for some $z, t \in A_{\bar{1}}$, then R_t is not invertible and hence $R_u R_t = L_u L_t = 0$ for any $u \in A_{\bar{1}}$. Thus $V A_{\bar{1}} R_t = 0 = (A_{\bar{1}} V) L_t$ and $t = 0$. We proved that for any $0 \neq x \in A_{\bar{1}}$ the operator R_x is invertible on V. In particular, we have $V_{\bar{1}} = V_{\bar{0}} x$, $V_{\bar{0}} = V_{\bar{1}} x$.

Let below $v \in V$, $x, y \in A_{\bar{1}}$, $a, b, c \in A_{\bar{0}}$. From super-linearized identity (10) we have

$$(vx)(ay) \pm (yx)(av) = (v(xa)y \pm (y(xa))v,$$

which gives $(vx)(ay) = (v(xa))y$ or

$$R_x R_{ay} = R_{xa} R_y. \tag{12}$$

For $y = x$ we have $R_x R_{ax} = R_x R_{xa}$, which by invertibility of R_x implies $R_{xa} = R_{ax}$ or $R_{[a,x]} = 0$, and eventually $[a, x] = 0$, that is,

$$[A_{\bar{0}}, A_{\bar{1}}] = 0. \tag{13}$$

Furthermore, applying (12) and (13), we get

$$R_{(ax)b} R_x = R_{ax} R_{bx} = R_{ax} R_{xb} = R_{xb} R_{ax} = R_{(xb)a} R_x = R_{a(xb)} R_x,$$

which implies $R_{(a,x,b)} = 0$ and therefore

$$(A_{\bar{0}}, A_{\bar{1}}, A_{\bar{0}}) = 0. \tag{14}$$

Let R be the F-subalgebra of $\mathcal{Z}_0$ generated the elements $\{R_x R_y \mid x, y \in A_{\bar{1}}\}$, then $R \subseteq \Gamma_{\bar{0}}$ is a commutative domain. Denote by K the field of fractions of R. Fix $0 \neq x \in A_{\bar{1}}$ and let $\alpha = (R_x)^2$. Then for any $y \in A_{\bar{1}}$, $R_y = (R_x R_y)\alpha^{-1} R_x$. Consider the F-linear mapping $\tau : A_{\bar{1}} \to K$ given by $\tau(y) = (R_x R_y)\alpha^{-1}$, then $R_y = \tau(y) R_x$. Clearly, $\ker \tau = 0$, hence $T = \tau(A_{\bar{1}})$ is an F-vector subspace of K isomorphic to $A_{\bar{1}}$. Furthermore, define an F-linear mapping $\sigma : A_{\bar{0}} \to K$ by $\sigma(a) = \tau(ax)$ and denote $S = \sigma(A_{\bar{0}})$.

Lemma 5.1. *The mappings τ and σ satisfy the conditions:*

$$i) \quad \tau(ay) = \sigma(a)\tau(y), \quad ii)\, \sigma(ab) = \sigma(a)\sigma(b),$$
$$iii) \qquad \ker \sigma = 0, \qquad iv)\, L_y = \tau(y)L_x,$$

for all $a, b \in A_{\bar{0}}, \ y \in A_{\bar{1}}$.

Proof. We have

$$\begin{aligned}
\sigma(a)\tau(y) &= (R_x R_{ax}\alpha^{-1})(R_x R_y \alpha^{-1}) = R_{ax} R_y R_x R_x \alpha^{-2} \\
&= R_{ax} R_y \alpha^{-1} \overset{(12)}{=} R_x R_{ay} \alpha^{-1} = \tau(ay),
\end{aligned}$$

proving $i)$. Furthermore,

$$\begin{aligned}
\sigma(a)\sigma(b) &= \tau(ax)\tau(bx) = (R_x R_{ax}\alpha^{-1})(R_x R_{bx}\alpha^{-1}) = R_{ax} R_{bx} \alpha^{-1} \\
&\overset{(12)}{=} R_x R_{a(bx)}\alpha^{-1} \overset{(14)}{=} R_x R_{(ab)x}\alpha^{-1} = \sigma(ab),
\end{aligned}$$

which proves $ii)$. Let now $a \in \ker \sigma$, then by $i)$, $\tau(ay) = 0$ for any $y \in A_{\bar{1}}$ and $aA_{\bar{1}} = 0$. By Lemma 3.1, (V, a, x) is a subbimodule of V. If $(V, a, x) = 0$ then we have $R_a R_x = 0$ and $R_a = 0$. Similarly, $L_a = 0$ and $a \in \operatorname{Ann} V = 0$. If $(V, a, x) = V$ then $Va = (V, a, ax) = 0$ and $aV = 0$, and again $a = 0$.

To prove $iv)$, consider

$$\begin{aligned}
L_y &= L_y \alpha \alpha^{-1} = L_y R_x R_x \alpha^{-1} = L_y L_x L_x \alpha^{-1} \\
&= R_y R_x L_x \alpha^{-1} = \tau(y)L_x;
\end{aligned}$$

$\square$

Corollary 5.1. *Let $B = K \oplus Ku$, $u^2 = 0$, with $B_{\bar{0}} = K$, $B_{\bar{1}} = Ku$, then the application $a_0 + y_1 \mapsto \sigma(a_0) + \tau(y_1)u$ is an isomorphism of A and $S + Tu \subseteq B$. In particular, $A_{\bar{0}}$ is an associative and commutative domain.*

Lemma 5.2. *For any $a \in A_{\bar{0}}$, $s \in A_{\bar{0}} \cup A_{\bar{1}}$ hold*

$$i) \; R_a = L_a \in \mathcal{Z}, \quad ii) \; R_a R_s = R_{as}.$$

Proof. By Lemma 3.1, (V, a, b) is a subbimodule of V. If it is not zero then by (4) we have $V = (V, a, b) = ((V, a, b), a, b) = [a, b](V, a, b) = 0$, a contradiction. Therefore $R_a R_b = R_{ij} = R_{ba} = R_b R_a$.

Furthermore, in view of (2), (13) and (14), (V, a, x) is a subbimodule of V for any $x \in A_{\bar{1}}$. If $(V, a, x) = 0$ then $R_a R_x = R_{ax} = R_{xa} = R_x R_a$. Moreover, in this case we have $\sigma(a) R_x = R_{ax} = R_a R_x$, implying $R_a = \sigma(a)$. Similarly $L_a = \sigma(a)$ and $L_a = R_a \in \mathcal{Z}$.

Assume that $(V, a, x) = V$. Then for any $v \in V$ we have $(v, a, x)a = (v, a, ax) = (v, a, xa) = a(v, a, x)$, that is, $R_a = L_a$. Furthermore, we have $R_a \circ R_x = 2R_{ax}$, $[R_a, R_x] = [L_a, R_x] = -R_a R_x + R_{ax}$. Summing the two equations we get $3R_a R_x = 3R_{ax}$. By (2), $R_a R_x - R_{ax} \in \Gamma_{\bar{1}} \subseteq \mathcal{Z}_{\bar{1}}$. Since Γ is a supercommutative domain, $\Gamma_{\bar{1}}$ may be non-zero only if char $F = 2$. Hence we have $R_a R_x = R_{ax}$. $\qquad \square$

Lemma 5.3. *If $cha F \neq 3$ then $L_x R_x \notin \mathcal{Z}$. If $L_x R_x \in \mathcal{Z}$ then char $F = 3$ and $R_x = L_x$.*

Proof. Assume that $L_x R_x \in \mathcal{Z}$, then we have $L_x R_x R_x = R_x L_x R_x$ which implies $L_x R_x = R_x L_x$. By (11), $2L_x R_x = [L_x, R_x]_s = -(R_x)^2 = -(L_x)^2$. Therefore, $2L_x = -R_x$, $2R_x = -L_x$, implying $L_x = R_x$. Since $x \in A_{\bar{1}}$ is arbitrary, V is supercommutative, and we have

$$3vR_x R_y = 3(v, x, y) = 3(x, y, v) \overset{(1)}{=} [xy, v] - x[y, v] - (-1)^{|y||v|}[x, v]y = 0,$$

which is impossible if char $F \neq 3$. If char $F = 3$ then we have $(L_x R_x - \alpha)^2 = -\alpha^2 - \alpha L_x R_x - 2\alpha L_x R_x + \alpha^2 = 0$ (see Section 5.1). Therefore, if $L_x R_x \in \mathcal{Z}$ then $L_x R_x = \alpha = R_x R_x$, and $R_x = L_x$. $\qquad \square$

Return to the proof of the Theorem. Assume that $L_x R_x \notin \mathcal{Z}$ then there exists $v \in V_{\bar{0}}$ such that v and $w = vL_x R_x$ are linearly independent over the division algebra $\mathcal{Z}_{\bar{0}}$. For $y, z \in A_{\bar{1}}$ we have $L_y R_z = (R_y R_z \alpha^{-1})(L_x R_x) \in R[\alpha^{-1}]L_x R_x$. Since V is irreducible, it follows from Lemma 5.2 and (11) that $vR + wR[\alpha^{-1}] = V_{\bar{0}}$. Thus we have

$$V_{\bar{0}} = vR + wR[\alpha^{-1}] \subseteq vK + wK \subseteq v\mathcal{Z}_{\bar{0}} + w\mathcal{Z}_{\bar{0}} \subseteq V_{\bar{0}},$$

which implies that $R = K = \mathcal{Z}_{\bar{0}}$.

If $L_x R_x \in \mathcal{Z}$ then $\operatorname{char} F = 3$, $L_x = R_x$, and for any $0 \neq v \in V_{\bar{0}}$ we have $V_{\bar{0}} = vR$, which implies as above $R = K = \mathcal{Z}_{\bar{0}}$. Notice that $R_y R_z = \tau(y)\tau(z)\alpha \in T^2\alpha$, hence $K = \operatorname{alg}\langle T^2\alpha\rangle$.

Define a unital K-linear action of B on V by setting $v \cdot u = vR_x$, $u \cdot v = vL_x$, then evidently V is an alternative irreducible faithful B-bimodule. The restriction of this action on $S + Tu \cong A$ coincides with the A-bimodule structure of V.

Conversely, let K be a field extension of F, and let T, S and α be an F-subspace, F-subalgebra and F-algebraic element of K such that $ST + TS \subseteq T$ and $\operatorname{alg}_F\langle \alpha T^2\rangle = K$. Consider the superalgebra $B = K \oplus Ku$, $u^2 = 0$, then $A = S + Tu$ is a subsuperalgebra of B. Let V be an irreducible unital faithful alternative B-superbimodule such that $\rho(u)^2 = \alpha$, and assume that $W \subseteq V$ is an A-subsuperbimodule of V. We have for any $w \in W$, $t_1, t_2 \in T$: $(w \cdot t_1 u) \cdot t_2 u = (t_1 t_2)(w \cdot u) \cdot u = (\alpha t_1 t_2)w \in W$, hence W is a K subspace of V. Let $0 \neq t \in T$, then $W \supseteq t^{-1}W \cdot (tu) = W \cdot u$, and $W = V$. $\qquad\square$

6 Bimodules over the simple superalgebras.

6.1 Reduction of prime superalgebras to simple ones

We start this section with reduction of the case of prime associative superalgebras to the simple finite dimensional ones.

We will say that a superalgebra A with the center $Z(A) = Z_{\bar{0}} \oplus Z_{\bar{1}}$ is *an even central order* in a superalgebra B if $B = (Z_{\bar{0}}^*)^{-1}A$. In this case we will also call B to be *the even central closure* of A.

Lemma 6.1. *Let A be a prime associative superalgebra and let V be a faithful irreducible alternative A-superbimodule which is not associative. Then $Z_{\bar{0}} = Z(A)_{\bar{0}} \neq 0$, and the even central closure $B = (Z_{\bar{0}}^*)^{-1}A$ is a simple superalgebra which is finite dimensional over the even part of its supercenter $(Z_{\bar{0}}^*)^{-1}Z$. Moreover, $(Z_{\bar{0}}^*)^{-1}Z \subseteq \mathcal{Z}(V)$, and V has a natural structure of a B-bimodule.*

Proof. Since A is associative, we can repeat the arguments from Section 4 and to get $V_{as} := \{v \in V \mid (v, A, A) = 0\} = 0$ and $V \cdot [N, A_{\bar{0}}] = 0$, where N is the associative center of the split null extension $E = A \oplus V$. Therefore $A_{\bar{0}}$ satisfies the identity $[[x, y]^4, z] = 0$. By [1], A is also a PI-algebra.

Consider the quotient algebra $A/\mathcal{B}(\mathcal{A})$, where $\mathcal{B}(\mathcal{A})$ is the Baer radical of the algebra A. Since $A/\mathcal{B}(\mathcal{A})$ is semiprime (as algebra) and PI, it follows

that there exists a nontrivial central polynomial $g(x_1, \ldots, x_n)$ of $A/\mathcal{B}(\mathcal{A})$ [12]. Hence $[g(a_1, \ldots, a_n), A] \subseteq \mathcal{B}(\mathcal{A})$ for arbitrary homogeneous elements a_k. But the Baer radical of a prime superalgebra does not contain nonzero homogeneous elements (see [4]), therefore $[g(a_1, \ldots, a_n), A] = 0$ and $Z_{\bar{0}} \neq 0$.

Consider the prime superalgebra $B = (Z_{\bar{0}}^*)^{-1}A$. Let I be a nonzero graded ideal of B. Then I is a prime superalgebra as well, and $I \bigcap Z_{\bar{0}}(B) = Z_{\bar{0}}(I) \neq 0$. The proof of these statements repeats verbatim the non-graded case. Hence $I = B$ and B is a simple superalgebra. It is also clear that $Z_{\bar{0}}(B) = (Z_{\bar{0}}^*)^{-1}Z$ is a field.

Due to [23], B is simple as an algebra or there exists a proper (non-graded) ideal I in B such that $B = \pi_0(I) \oplus \pi_1(I)$, where $\pi_i : I \longrightarrow B_i$ are the natural projections. It is easy to see that in this case $B \cong B_{\bar{0}}[u]$ where $B_{\bar{0}}$ is simple.

In the first case, since B is a simple PI-algebra it follows that B is finite-dimensional over $Z(B)$. But $Z(B)$ is finite-dimensional over $Z_{\bar{0}}(B)$. Indeed, if $Z_{\bar{1}}(B) \neq 0$ then, fixing $0 \neq z_1 \in Z_{\bar{1}}(B)$, we have $z_1^2 = \alpha \neq 0 \in Z_{\bar{0}}(B)$ and then for any $x \in Z_{\bar{1}}(B)$ we get $x = x(\alpha^{-1}z_1^2) = (\alpha^{-1}xz_1)z_1 \in Z_{\bar{0}}(B) \cdot z_1$.

In the second case $B_{\bar{0}}$ is a simple PI-algebra, hence $B_{\bar{0}}$ is finite-dimensional over $Z(B_{\bar{0}})$. It is easily seen that B is finite-dimensional over $Z(B_0)$ too and, in this case, $Z(B) = Z(B_{\bar{0}})[u]$.

By Lemma 4.1, $Z(A)$ is contained in the centralizer $\mathcal{Z}(V)$ of the bimodule V, which is a graded division superalgebra. Therefore, $(Z_{\bar{0}}^*)^{-1}Z \subseteq \mathcal{Z}$ and V has a natural structure of a B-bimodule. Clearly, V is irreducible and faithful over B. $\square$

In non-associative case we have

Lemma 6.2. *Let A be a prime alternative non-associative non-trivial superalgebra of $\mathrm{char} \neq 3$ and let V be a faithful irreducible alternative A-superbimodule. Then $\mathrm{char}\, F = 2$, $Z_{\bar{0}} = Z(A)_{\bar{0}} \neq 0$, and the even central closure $B = (Z_{\bar{0}}^*)^{-1}A$ is a simple superalgebra of type $\mathbb{O}(4|4)$ or $\mathbb{O}[u]$. As above, $(Z_{\bar{0}}^*)^{-1}Z \subseteq \mathcal{Z}(V)$, and V has a natural structure of a B-bimodule.*

Proof. The first part of the Lemma follows from the classification of prime alternative superalgebras in [17]. The second part follows from Lemma 4.1 as above. $\square$

Therefore, except for the case of char 3, the classification of irreducible alternative non-associative superbimodules over prime superalgebras is reduced to bimodules over simple central finite-dimensional superalgebras, where we call a superalgebra A over a field F *central* if $Z(A)_{\bar{0}} = F$.

We will show next that, modulo the superalgebras of type $B = K[u]$, for the case of char $\neq 2$ it suffices to consider only non-associative superalgebras.

Lemma 6.3. *In the conditions of Lemma 6.1, if* $\operatorname{char} F \neq 2$ *then the even central closure* B *of* A *is of the type* $B = K + Ks$, $s^2 = \lambda \in K$, $\lambda \neq 0$, *and the structure of* V *and* A *is given by Theorem 5.1.*

Proof. By Lemma 6.1, $Z = Z_{\bar{0}}(A) \neq 0$, A is an even central order in a simple central superalgebra B over the field $K = (Z_{\bar{0}}^*)^{-1} Z_{\bar{0}}$, and V is a faithful irreducible alternative bimodule over B. Let $\tilde{K}$ be the algebraic closure of K; then $\tilde{B} = \tilde{K} \otimes B$ is a simple central superalgebra over $\tilde{K}$ and $\tilde{V} = \tilde{K} \otimes V$ is an alternative unital bimodule over $\tilde{B}$. If $\dim_{\tilde{K}} \tilde{B} > 2$, then by [9] $\tilde{V}$ is completely reducible and all its irreducible components are associative. Hence in this case V is associative, contrary to the assumption. Therefore, $\dim_{\tilde{K}} \tilde{B} = \dim_K B = 2$ and $B = K + Ks$, $s^2 = \lambda \in K^*$. $\qquad\square$

Remark 6.1. *Observe that, contrary to the case of algebras in Section 4, A may be not simple and hence in general $B \neq A$. As an example, consider the superalgebra* $B = \mathbb{Q}[\sqrt{1}] = \mathbb{Q} \oplus \mathbb{Q}u$, $u^2 = 1$ *and let* $V = \mathbb{Q}v_0 \oplus \mathbb{Q}v_1$ *be the irreducible supermodule of type 3 from Theorem 2.2 for* $\alpha = -\frac{3}{7}$, $\delta = \frac{8}{7}$. *Let* $A = \mathbb{Q}_7[\sqrt{1}] = \mathbb{Q}_7 \oplus \mathbb{Q}_7 u$, $u^2 = 1$, *where* $\mathbb{Q}_7$ *stands for the subring of rational numbers whose denominators are not divisible by 7. Then V is an irreducible faithful alternative A-bimodule, and* $B = (Z_{\bar{0}}^*)^{-1} A \neq A$.

If one wanted A to be an algebra over a field and not just a ring, then one may take in this example $A = F[[x]][\sqrt{1}]$, $B = F((x))[\sqrt{1}]$, $\alpha = \frac{-x^2+1}{x^2+x+1}$, $\delta = x(1 + \alpha)$ where F is a field, $F[[x]]$ and $F((x))$ are the algebra of formal series and the field of formal Laurent series over F.

6.2 Superbimodules in characteristic 2.

In this section, we will classify irreducible alternative non-associative superbimodules of characteristic 2.

Recall that every $\mathbb{Z}_2$-graded alternative algebra of characteristic 2 is an alternative superalgebra. The simple superalgebras $\mathbb{O}(4|4)$ and $\mathbb{O}[u]$ provide us the irreducible superbimodules $\operatorname{Reg} \mathbb{O}(4|4)$, $\operatorname{Reg}(\mathbb{O}[u])$.

Furthermore, consider a generalized quaternion algebra $\mathbb{H}$ with the $\mathbb{Z}_2$-grading coming from the Cayley-Dickson process; then it is an alternative superalgebra which we denote as $\mathbb{H}(2|2)$. As in the non-graded case, the category of $\mathbb{Z}_2$-graded Cayley bimodules over $\mathbb{H}(2|2)$ is isomorphic to the category of (graded) left $\mathbb{H}(2|2)$-modules. For a left $\mathbb{H}(2|2)$-module L we again denote by $\operatorname{Cay} L$ the corresponding Cayley $\mathbb{H}(2|2)$-bimodule. Then $\operatorname{Cay} L$ is an alternative superbimodule over $\mathbb{H}(2|2)$ in the characteristic 2 case.

Finally, let A be an alternative algebra and V be an A-bimodule; consider the split null extension $E = E(A, V)$. The double $E[u] = E + Eu$, $u^2 = \alpha \neq 0$ is an alternative superalgebra with the even part E and the odd part Eu (recall that char $F = 2$). Moreover, the ideal $V[u] = V + Vu$ may be considered as a bimodule over the alternative superalgebra $A[u] = A + Au$, and if V is irreducible then so is $V[u]$. We will denote this bimodule as $V[u]$ or as $V[\sqrt{1}]$ when $u^2 = 1$.

Theorem 6.1. *Let V be an irreducible faithful alternative non-associative bimodule over a nontrivial prime alternative superalgebra A of characteristic 2 with $\dim_{(Z^*)^{-1}Z}(Z^*)^{-1}A > 2$. Then, up to the graded isomorphism, we have the following possibilities:*

- *$A = \mathbb{H}(2|2)$, $V = \operatorname{Cay} L$, L is a minimal $\mathbb{Z}_2$-graded left ideal of $\mathbb{H}(2|2)$;*

- *$A = \mathbb{H}[u]$, $V = (\operatorname{Cay} L)[u]$, L is a minimal left ideal of $\mathbb{H}$;*

- *$A = \mathbb{O}(4|4)$, $V = \operatorname{Reg}(\mathbb{O}(4|4))$;*

- *$A = \mathbb{O}[u]$, $V = \operatorname{Reg}(\mathbb{O}[u])$.*

Proof. Consider first the associative case. By Lemma 6.1, $B = (Z^*)^{-1}A$ is a simple central superalgebra over the field $K = (Z^*)^{-1}Z$. Let $\tilde{K}$ be the algebraic closure of K, then $\tilde{B} = \tilde{K} \otimes_K B$ is a simple central superalgebra over $\tilde{K}$ and by [23] $\tilde{B}$ is isomorphic to $M_{m|k}(\tilde{K})$, $1 \leq k \leq m$, or to $M_n(\tilde{K})[\sqrt{1}]$, $n > 1$. Let us show that it should necessary hold $m + k = n = 2$. Consider $\tilde{V} = \tilde{K} \otimes_K V$; it is a bimodule over $\tilde{B}$ and by [27, Lemma 7.5] we have $[N, \tilde{B}]_s \subseteq \operatorname{Ann}(\tilde{V}, \tilde{B}, \tilde{B})$, where N is the associative center of the split null extension $E(\tilde{B}, \tilde{V})$. Notice that since B is associative and V is irreducible, we have $V = (V, B, B)$. Therefore $V[N, \tilde{B}]_s = 0$, which implies easily that $[N, \tilde{B}]_s = 0$. In particular, the following inclusion holds in $\tilde{B}$:

$$[[a, x] \circ_s [a, y], a] \in Z_{\bar{0}}(\tilde{B}) = \tilde{K} \tag{15}$$

for any $a \in \tilde{B}_{\bar{0}}$, $x, y \in \tilde{B}_{\bar{1}}$. In fact, by [27, (7.25) and Lemma 7.1], we have the inclusion $[[a, b]^2, a] \in N(A)$ in any alternative algebra A. Superlinearizing this inclusion on b, we get the inclusion

$$[[a, x] \circ_s [a, y], a] \in N(E), \tag{16}$$

for any $a \in E_{\bar{0}}$, $x, y \in E_{\bar{0}} \cup E_{\bar{1}}$, which implies (15). Assume now that $\tilde{B} = M_{m|k}(\tilde{K})$ with $m > 1$ and substitute in (15) $a = e_{11} + e_{22} + e_{12}$, $x = e_{1,m+1}$, $y =$

$e_{m+1,1}$. We obtain

$$
\begin{aligned}
[[a, x] \circ_s [a, y], a] &= [[e_{1,m+1}, e_{m+1,1} + e_{m+1,2}], a] \\
&= [e_{11} + e_{12} + e_{m+1,m+1}, e_{11} + e_{22} + e_{12}] = e_{12} \notin \tilde{K},
\end{aligned}
$$

a contradiction. Similarly, assume that $\tilde{B} = M_n(\tilde{K})[\sqrt{1}]$ with $n > 2$ and substitute in (15) $a = e_{11} + e_{22} + e_{12}$, $x = e_{13}u$, $y = e_{31}u$:

$$
\begin{aligned}
[[a, x] \circ_s [a, y], a] &= [[ue_{13}, u(e_{31} + e_{32})], a] \\
&= [e_{11} + e_{12} + e_{33}, e_{11} + e_{22} + e_{12}] = e_{12} \notin \tilde{K},
\end{aligned}
$$

a contradiction again.

Returning to the superalgebra B we conclude that in the first case B is a generalized quaternion superalgebra $\mathbb{H}(2|2)$ and in the second case $B_{\bar{0}} \cong \mathbb{H}$.

Let $B = \mathbb{H}(2|2)$, $B_{\bar{0}} = K + Ki$, $B_{\bar{1}} = Kj + Kk$, $i^2 = i + \alpha$, $k = ij = j(i+1)$, $j^2 = \beta$, where $\alpha, \beta \in K$, $(4\alpha + 1)\beta \neq 0$ (see [27, Chapter 2]). Substituting in (16) $a = i$, $x = j$, $y = v \in V$, we get

$$
([[i, j] \circ_s [i, v], i], B, B) = ([[[v, i], j], i], B, B) = 0,
$$

that is, $[[[V, i], j], i] \subseteq V_{as} = \{v \in V \mid (v, B, B) = 0\}$ which is a subbimodule of V. Since V is nonassociative, $V_{as} = 0$, and we have $[[[V, i], j], i] = 0$. Let $W = \{w \in V \mid [w, i] = 0\}$, show that $W = 0$. For any $w \in W$, $b \in B$ we have

$$
(w, b, i) = (w, b, i^2 - \alpha) = (w, b, i^2) = (w \circ i, b, i) = ([w, i], b, i) = 0,
$$

that is, $(W, B, i) = 0$. Further, $(W, B, B) \circ i \subseteq (W \circ i, B, B) + W \circ (i, B, B) = 0$, thus $(W, B, B \circ i) \subseteq (W, B, B) \circ i + (W, B, i) \circ B = 0$. Observe that $j = j \circ i$, $k = k \circ i$, which implies that $(W, B, B) = 0$ and $W \subseteq V_{as} = 0$.

Therefore, we have $[[V, i], j] = 0$ and similarly $[[V, i], k] = 0$. Let us prove that $U = [V, i] = V \circ i$ is a subbimodule of V. Clearly, $Ui + iU \subseteq U$. Note that $(V, B, i) = (V, B, i^2) = (V, B, i) \circ i \subseteq U$. We have also $[U, j] = [U, k] = 0$. Consider

$$
\begin{aligned}
[v, i]j &= [v, i^2]j = [v \circ i, i]j = [v \circ i, ij] - i[v \circ i, j] + 3(v \circ i, i, j) \\
&= [v \circ i, k] + (v \circ i, i, j) = (v \circ i, i, j) \in (V, B, i) \subseteq U.
\end{aligned}
$$

Thus $Uj \subseteq U$ and similarly $Uk \subseteq U$, hence U is a subbimodule of V. We have already showed that the mapping $v \mapsto v \circ i$ is injective. Therefore $U = V$, and we have $[V, j] = [V, k] = 0$. Consider $3(v, j, j) = [vj, j] - v[j, j] + [v, j]j = 0$.

Similarly, we have $(V, k, k) = 0$, which proves that V is an alternative $\mathbb{Z}_2$-graded B-bimodule.

Denote $T = \{v - [v, i] \,|\, v \in V\}$, then clearly $Ti + iT \subseteq T$. Furthermore,

$$
\begin{aligned}
[v, i]j &= [v, ij] - i[v, j] + 3(v, i, j) = 3(v, i, j) \\
&= [vi, j] - v[i, j] - [v, j]i = v[i, j] = vj,
\end{aligned}
$$

and similarly $[v, i]k = vk$. Thus $Tj = Tk = 0$ and T is a subbimodule of V. Since V is faithful, $T = 0$ and we have $v \circ i = v$ for any $v \in V$. Therefore, $bv = v\bar{b}$ for any $b \in B$, and V is a Cayley bimodule over B. Since V is irreducible, $V = \operatorname{Cay} L$, where L is an irreducible left $\mathbb{H}(2|2)$-module. It is isomorphic to a minimal graded left ideal of $B = \mathbb{H}(2|2)$. As at the end of the proof of Theorem 4.1, we get that L is an irreducible graded left A-module and A is a graded primitive PI-algebra. By the graded version of Kaplansky's theorem, A is graded simple. Hence $Z(A)$ is a graded division algebra, $K = Z_{\bar{0}}$ and $A = B = \mathbb{H}(2|2)$.

Consider now the case when $\tilde{B} = M_2(\tilde{K})[\sqrt{1}] = M_2(\tilde{K}) + M_2(\tilde{K})u$. We have $Z_{\bar{1}}(\tilde{B}) = \tilde{K}u \neq 0$, therefore $Z_{\bar{1}}(B) \neq 0$ and B is not simple as an algebra. Consequently $B = B_{\bar{0}}[u] = \mathbb{H}[u] = \mathbb{H} + \mathbb{H}u, u^2 = \alpha \neq 0$. Take $a \in \mathbb{H}$ and consider $W = (V, a, u)$. It follows from (2) that W is a subbimodule of V. Assume that $W = V$, then $V = ((V, a, u), a, u)$. But we have for any $v \in V$ by super-linearized (4)

$$
((v, a, u), a, u) = ((u, a, u), a, v) + [a, u](v, a, u) + [a, v](u, a, u) = 0,
$$

a contradiction. Therefore $(V, \mathbb{H}, u) = 0$, and by (3) $[u, (V, \mathbb{H}, \mathbb{H})] \subseteq (u, V, \mathbb{H}) = 0$. Let us show that if $[u, v] = 0$ then $(u, u, v) = 0$. In fact, by (1), $3(u, u, v) = [u^2, v] - u[u, v] - [u, v]u = 0$. Thus $((V, \mathbb{H}, \mathbb{H}), u, u) = 0$.

Consider now the space (V, u, u); it is also a subbimodule of V. Assume that $(V, u, u) = V$, then we have

$$
(V, \mathbb{H}, \mathbb{H}) = ((V, u, u), \mathbb{H}, \mathbb{H}) = ((V, \mathbb{H}, \mathbb{H}), u, u) = 0.
$$

We have already seen that $(V, \mathbb{H}, u) = 0$ as well. Take $a, b \in \mathbb{H}$ and consider

$$
(v, a, bu) \overset{(8)}{=} (v, u, ba) + u(v, a, b) + a(v, u, b) = 0.
$$

Therefore $(V, \mathbb{H}, B) = 0$, which imlies by (2) that $[V, \mathbb{H}] \subseteq V_{as} = 0$. Consequently, V is an associative and commutative $\mathbb{H}$-bimodule. But it is impossible since for any $a, b \in \mathbb{H}$ and $v \in V$ we would have

$$
(ab)v = a(bv) = a(vb) = (vb)a = v(ba) = (ba)v, \quad [a, b]v = v[a, b] = 0,
$$

and $[a, b] = 0$, a contradiction. Therefore, $(V, u, u) = 0$.

It easy now to see that $(V, u, B) = 0$ and then $[V, u] \subseteq V_{as} = 0$. Therefore, $u \in Z(E(V, B))$ and $V = V_{\bar{0}} + V_{\bar{0}} u = V_{\bar{0}}[u]$. Clearly, $V_{\bar{0}}$ is an irreducible alternative $\mathbb{H}$-bimodule. If $V_{\bar{0}}$ were associative then V would be associative as well, which is not the case. Therefore, $V \cong \mathrm{Cay}\, L$ where L is a minimal left ideal of $\mathbb{H}$. As in the previous case, $A_{\bar{0}}$ is a primitive PI-algebra and $Z(A_{\bar{0}}) = Z(A)_{\bar{0}}$ is a field. Therefore $A = B$.

Assume now that $B = \mathbb{O}[u] = \mathbb{O} + \mathbb{O}u$, $u^2 = \lambda \in K$, $\lambda \neq 0$, where $\mathbb{O}$ is a central Cayley-Dickson algebra over K. As above, we have $(V, u, \mathbb{O}) = [u, (V, \mathbb{O}, \mathbb{O})] = ((V, \mathbb{O}, \mathbb{O}), u, u) = 0$. Assume that $(V, u, u) = V$, then $(V, \mathbb{O}, \mathbb{O}) = 0$ and V is a unital associative bimodule over $\mathbb{O}$, which is impossible. Hence $(V, u, u) = 0$. As above, we have $(V, u, \mathbb{O}) = [V, u] = 0$, hence $u \in Z_{\bar{1}}(E(V, B))$ and $V = V_{\bar{0}}[u]$. It is clear now that $V_{\bar{0}} \cong \mathrm{Reg}\, \mathbb{O}$. As in the proof of Theorem 4.1, we have also $A = B$.

Let finally $B = \mathbb{O}(4|4)$. Then $B_{\bar{0}} = \mathbb{H}$ and $B_{\bar{1}}$ is a Cayley bimodule over $\mathbb{H}$. By [17, Lemma 12], $V = V_a \oplus V_c$, where V_a is an associative $\mathbb{H}$-bimodule and V_c is a Cayley $\mathbb{H}$-bimodule. Moreover, by [7, Lemma 3.2], we have $V_a B_{\bar{1}} + B_{\bar{1}} V_a \subseteq V_c$, $V_c B_{\bar{1}} + B_{\bar{1}} V_c \subseteq V_a$. Since V is faithful, this implies that $V_a \neq 0$, $V_c \neq 0$. Let $Z_a = \{v \in V_a \,|\, [v, \mathbb{H}] = 0\}$. As in the proof of Lemma 3.3 in [7] we have that $Z_a \neq 0$ and $[Z_a, B] = 0$. Furthermore, $3(Z_a, B, B) \subseteq [BB, Z_a] + B[B, Z_a] + [B, Z_a]B = 0$. Hence $0 \neq Z_a \subseteq Z(E(V, B))$. Choose some homogeneous element $0 \neq u \in Z_a$, then the subspace $u \cdot B$ is a B-subbimodule of V and the mapping $\varphi : a \mapsto u \cdot a$ is a B-bimodule homomorphism of $\mathrm{Reg}\, B$ onto uB, in case when u is even, or of $(\mathrm{Reg}\, B)^{op}$ onto uB, in case when u is odd. Since both $\mathrm{Reg}\, B$ and $(\mathrm{Reg}\, B)^{op}$ are irreducible, and $\varphi(1) = u \neq 0$, we have that $uB = V$ is isomorphic to $\mathrm{Reg}\, B$ or to $(\mathrm{Reg}\, B)^{op}$. Clearly, in both cases we have $A = B$. $\qquad\square$

6.3   Superbimodules over the superalgebra $B(\Gamma, D, \gamma)$.

In this section we consider the last remained case of superbimodules in characteristic 3. Unlike the other cases, we can not classify irreducible alternative superbimodules of characteristic 3 in any dimension, since the prime alternative superalgebras of characteristic 3 are classified only under certain restrictions.

Nevertheless, due to [8], every finite dimensional non-associative prime alternative superalgebra A is simple. Thus, if $\mathrm{char}\, A = 3$ then A is one of the superalgebras $B(1|2)$, $B(4|2)$, $B(\Gamma, D, \gamma)$. The supermodules over the su-

peralgebras $B(1|2)$, $B(4|2)$ were classified in [7, 21] (see Theorems 2.3, 2.4). Therefore, to finish the classification of irreducible finite dimensional super-bimodules, it suffices to classify them over the superalgebra $B(\Gamma, D, \gamma)$. We will do this for any dimension.

Below V denote an irreducible faithful bimodule over the superalgebra $B = B(\Gamma, D, \gamma)$ which is not associative.

We start with the studying the structure of V as a bimodule over $B_{\bar{0}} = \Gamma$.

Lemma 6.4. $(V, \Gamma, \Gamma) = [V, \Gamma] = 0$.

Proof. Fix $a, b \in \Gamma$. Since Γ is commutative and $(B, \Gamma, \Gamma) = 0$, by (2) it follows that (V, a, b) is a subbimodule of V. If $(V, a, b) = V$, then $V = (V, a, b) = ((V, a, b), a, b) = 0$ by (4). Thus, $(V, a, b) = 0$.

Consider $[V, a]$. Since $[\Gamma, B] = 0$ and char $B = 3$, it follows from (1) that $[V, a]$ is a subbimodule of V. Hence, $[V, a] = 0$ or $[V, a] = V$.

Note that $D(a^3) = 3a^2 D(a) = 0$. We will show next that if $D(b) = 0$ for some $b \in \Gamma$ then $[V, b] = 0$.

Let $c, d \in \Gamma$. We have

$$
\begin{aligned}
(b, \bar{c}, \bar{d}) &= \bar{b}c \cdot \bar{d} - b \cdot (\gamma cd + 2D(c)d + cD(d)) \\
&= \gamma bcd + 2D(bc)d + bcD(d) - \gamma bcd - 2bD(c)d - bcD(d) \\
&= 2D(bc)d - 2bD(c)d = 0.
\end{aligned}
$$

It follows now from (2) that (V, b, x) is a subbimodule of V for any $x \in \Gamma$. If $(V, b, x) = V$ then $V = (V, b, x) = ((V, b, x), b, x) = 0$, by superized (6). Thus $(V, b, \bar{\Gamma}) = 0$.

If $[V, b] = V$ then again from (2) we get

$$
(V, B, B) = ([V, b], B, B) \subseteq (BB, V, b) + B(B, V, b) + (B, V, b)B = 0,
$$

which contradicts to the nonassociativity of V. Hence, $[V, b] = 0$.

Now, if $[V, a] = V$, then $V = [V, a] = [[V, a], a] = [[[V, a], a], a] = [V, a^3] = 0$, by the above. Hence, $[V, a] = 0$. $\square$

The next proposition is an adaptation of the results from [17] to the bimodule situation.

Proposition 6.1. *Let* $V = V_{\bar{0}} \oplus V_{\bar{1}}$ *be an irreducible bimodule over an alternative superalgebra* $A = A_{\bar{0}} \oplus A_{\bar{1}}$. *If* N *is a proper* $A_{\bar{0}}$-*subbimodule of* V_i, *and* $(N, A_{\bar{1}}, A_{\bar{1}}) \subseteq N$, *then* $V = (N \oplus A_{\bar{1}} N A_{\bar{1}}) \oplus (N A_{\bar{1}} \oplus A_{\bar{1}} N)$. *Moreover, if* $[N, A_{\bar{1}}]_s = 0$ *then* $N = V_i$.

Proof. It is clear that $(N+A_{\bar{1}}NA_{\bar{1}})\oplus(NA_{\bar{1}}+A_{\bar{1}}N)$ is a proper subbimodule of V and hence it is equal to V. Let us prove that the sums in the summands are direct. Suppose that $N \cap A_{\bar{1}}NA_{\bar{1}} = L \neq 0$. Then $L + L \circ A_{\bar{1}}$ is a nonzero subbimodule of V. Hence, $L + L \circ A_{\bar{1}} = V$, but this contradicts to the fact that L is a proper $A_{\bar{0}}$-subbimodule of V_i. Therefore, $N \cap A_{\bar{1}}NA_{\bar{1}} = 0$.

Now let $L = NA_{\bar{1}} \cap A_{\bar{1}}N$. Then $LA_{\bar{0}} \subseteq NA_{\bar{1}} \cdot A_{\bar{0}} \subseteq NA_{\bar{1}}$. On the other hand, $LA_{\bar{0}} \subseteq A_{\bar{1}}N \cdot A_{\bar{0}} \subseteq (A_{\bar{1}}, N, A_{\bar{0}}) + A_{\bar{1}}N \subseteq A_{\bar{1}}N$. Hence, $LA_{\bar{0}} \subseteq L$. Similarly, $A_{\bar{0}}L \subseteq L$. Thus $L + L \circ A_{\bar{1}} = V$. But $LA_{\bar{1}} \subseteq NA_{\bar{1}} \cdot A_{\bar{1}} \subseteq N$ and, similarly, $A_{\bar{1}}L \subseteq N$. Therefore $V = L + N$, a contradiction. So, $NA_{\bar{1}} \cap A_{\bar{1}}N = 0$.

Finally, assume that $[N, A_{\bar{1}}] = 0$. Then we have

$$(A_{\bar{1}}N)A_{\bar{1}} \subseteq (NA_{\bar{1}})A_{\bar{1}} \subseteq (N, A_{\bar{1}}, A_{\bar{1}}) + NA_{\bar{0}} \subseteq N,$$

therefore $V = N \oplus N \circ A_{\bar{1}}$ and $N = V_i$. $\qquad\square$

Lemma 6.5. $(V_{\bar{0}}, \Gamma, \bar{\Gamma}) = 0$ *or* $(V_{\bar{1}}, \Gamma, \bar{\Gamma}) = 0$.

Proof. Let $W = \left\{ v \in V \mid (v, \Gamma, \bar{\Gamma}) = 0 \right\}$. We show first that $W \neq 0$. Recall that $(\Gamma, \Gamma, \bar{\Gamma}) = 0$. By superization of (4), in view of Lemma 6.4 we have

$$
\begin{aligned}
((V, \Gamma, \bar{\Gamma}), \Gamma, \bar{\Gamma}) \;\subseteq\; & ((\Gamma, \Gamma, \bar{\Gamma}), V, \bar{\Gamma}) + ((V, \bar{\Gamma}, \bar{\Gamma}), \Gamma, \Gamma) \\
& + ((\Gamma, \bar{\Gamma}, \bar{\Gamma}), V, \Gamma) + [V, \bar{\Gamma}](\bar{\Gamma}, \Gamma, \Gamma) \subseteq (V, \Gamma, \Gamma) = 0.
\end{aligned}
$$

Therefore, $W \neq 0$. It follows from (2) that $W\Gamma \subseteq W$. Moreover, by (6) we have

$$((W, \bar{\Gamma}, \bar{\Gamma}), \Gamma, \bar{\Gamma}) \subseteq (W, \Gamma, \bar{\Gamma}) + (W, \bar{\Gamma}, \Gamma) = 0.$$

Consequently, $(W, \bar{\Gamma}, \bar{\Gamma}) \subseteq W$. It is clear that W is a graded subspace of V, and we see that the components W_i, $i = 0, 1$, satisfy the condition of Proposition 6.1. Consider $[W, \bar{\Gamma}]$. Since $\text{char } B = 3$, it follows from (1) that $[W, \bar{\Gamma}]$ is a Γ-submodule of V. By superization of (7) and (8),

$$
\begin{aligned}
([W, \bar{\Gamma}], \bar{\Gamma}, \bar{\Gamma}) \;\subseteq\; & ([\bar{\Gamma}, \bar{\Gamma}], \bar{\Gamma}, W) + [(\bar{\Gamma}, \bar{\Gamma}, \bar{\Gamma}), W] + [(\bar{\Gamma}, \bar{\Gamma}, W), \bar{\Gamma}] \\
\;\subseteq\; & (\Gamma, \bar{\Gamma}, W) + [W, \bar{\Gamma}] = [W, \bar{\Gamma}].
\end{aligned}
$$

Consequently, the subspaces $[W_i, \bar{\Gamma}]$, $i = 0, 1$, satisfy the condition of Proposition 6.1 as well.

Using (1) and Lemma 6.4, we obtain

$$(\bar{\Gamma}[W, \bar{\Gamma}])\bar{\Gamma} \subseteq ([W, \bar{\Gamma}]\bar{\Gamma})\bar{\Gamma} \subseteq [W, \bar{\Gamma}] + [W, \bar{\Gamma}]\Gamma \subseteq [W, \bar{\Gamma}].$$

Therefore, for every $i = 0, 1$, by Proposition 6.1 we have $[W_i, \bar{\Gamma}] = 0$ or $V = [W_i, \bar{\Gamma}]\bar{\Gamma} + [W_i, \bar{\Gamma}]$.

If $[W_i, \bar{\Gamma}] = 0$, then by Proposition 6.1 $W_i = V_i$ and $(V_i, \Gamma, \bar{\Gamma}) = 0$.

In the second case we have $V_{1-i} = [W_i, \bar{\Gamma}]$. Then by (9), (10) we have

$$
\begin{aligned}
(V_{1-i}, \bar{\Gamma}, \Gamma) &= ([W_i, \bar{\Gamma}], \bar{\Gamma}, \Gamma) \subseteq [(\bar{\Gamma}, \bar{\Gamma}, \Gamma), W] \\
&+ [(\bar{\Gamma}, \bar{\Gamma}, W), \Gamma] + ([\Gamma, \bar{\Gamma}], \bar{\Gamma}, W) \subseteq [V, \Gamma] = 0,
\end{aligned}
$$

hence $V_{1-i} \subseteq W$. $\qquad\square$

Lemma 6.6. *If* $(V_{\bar{0}}, \Gamma, \bar{\Gamma}) = 0$*, then* $[V_{\bar{0}}, \bar{\Gamma}] = 0$*.*

Proof. Similarly as it was done for W in the proof of Lemma 6.5, we can show that $[V_{\bar{0}}, \bar{\Gamma}] = 0$ or $V = [V_{\bar{0}}, \bar{\Gamma}]\bar{\Gamma} + [V_{\bar{0}}, \bar{\Gamma}]$. Consider the last case. As in the proof of Lemma 6.5 we obtain that $(V_{\bar{1}}, \Gamma, \bar{\Gamma}) = ([V_{\bar{0}}, \bar{\Gamma}], \Gamma, \bar{\Gamma}) = 0$, so $(V, \Gamma, \bar{\Gamma}) = 0$.

Note that for any nonzero $a \in \Gamma$ we have $Va = V$, hence $V = Va = (Va)a = Va^3$. Thus $a^3 \neq 0$. Since Γ is D-simple [17], it follows from [19] that Γ is a field.

By (2) we have

$$
0 = (V_{\bar{0}}, \Gamma, \bar{\Gamma}) = ([V_{\bar{0}}, \bar{\Gamma}]\bar{\Gamma}, \Gamma, \bar{\Gamma}) = [V_{\bar{0}}, \bar{\Gamma}](\bar{\Gamma}, \Gamma, \bar{\Gamma}).
$$

In particular, for any $a \in \Gamma$ we have $0 = [V_0, \bar{\Gamma}](a, \bar{1}, \bar{1}) = 2[V_0, \bar{\Gamma}]D(a)$. (It was proved in [11] that any D-simple algebra contains an identity element 1.) By choosing a such that $D(a) \neq 0$ we obtain that $[V_0, \bar{\Gamma}] = 0$. $\qquad\square$

Let (A, D) be an algebra with a derivation D. An A-bimodule (V, d) with a linear mapping $d \colon V \to V$ is called *a bimodule with derivation* or *a D-bimodule* over the algebra (A, D), if the linear mapping $D + d \colon a + v \mapsto D(a) + d(v)$ is a derivation of the split null extension $E = A \oplus V$. If, in addition, V has no proper d-invariant A-subbimodules, then (V, d) is called *a D-simple A-bimodule*.

Let (V, d) be an associative and commutative D-bimodule over an algebra with a derivation (Γ, D). Consider the split null extension $E = \Gamma \oplus V$ with the derivation $D + d$ and construct the alternative superalgebra $B(E, D+d, \gamma)$. We have

$$
B(E, D + d, \gamma) = (\Gamma \oplus V) \oplus \overline{\Gamma \oplus V} = (\Gamma \oplus \bar{\Gamma}) \oplus (V \oplus \bar{V}) = B \oplus W,
$$

where $W = V \oplus \bar{V}$ is an alternative bimodule over the superalgebra $B = B(\Gamma, D, \gamma)$. We denote this bimodule by $B(V, d, \gamma)$.

From the definition, it is easy to recover the explicit action of $B(\Gamma, D, \gamma)$ on $B(V, d, \gamma)$:

$$a \cdot v = v \cdot a = av, \quad \bar{a} \cdot v = v \cdot \bar{a} = a \cdot \bar{v} = \bar{v} \cdot a = \overline{av},$$
$$\bar{a} \cdot \bar{v} = 2D(a)v + ad(v) + \gamma av, \quad \bar{v} \cdot \bar{a} = 2d(v)a + vD(a) + \gamma va.$$

We are now ready to prove

Theorem 6.2. *Let V be an irreducible faithful nonassociative alternative bimodule over the simple superalgebra $B(\Gamma, D, \gamma)$. Then, up to the changing of parity, $(V_{\bar{0}}, d)$ is a D-simple faithful associative and commutative bimodule over the algebra with derivation (Γ, D), where $d(v) = -(v, \bar{1}, \bar{1})$ is a derivation of $V_{\bar{0}}$; $V_{\bar{1}} = V_{\bar{0}} \cdot \bar{1}$ is an isomorphic copy of $V_{\bar{0}}$ and $V \cong B(V_{\bar{0}}, d, \gamma)$.*

Proof. By Lemmas 6.4, 6.5, 6.6, we may assume, up to the changing of the parity, that $(V_{\bar{0}}, \Gamma, B) = [V_{\bar{0}}, B] = 0$. Let $d : V_{\bar{0}} \to V_{\bar{0}}, \ d : v \mapsto 2(v, \bar{1}, \bar{1})$, then it follows from (2) that $(V_{\bar{0}}, d)$ is an associative and commutative D-bimodule over (Γ, D). For $v \in V_{\bar{0}}$, denote $\bar{v} = v \cdot \bar{1}$, then we have

$$
\begin{aligned}
a\bar{v} &= a(\bar{1}v) = (a\bar{1})v = \bar{a}v = (\bar{1}a)v = \bar{1}(av) = \overline{av}, \\
\bar{a}\bar{v} &= \bar{a}(\bar{1}v) = (\bar{a}\bar{1})v - (\bar{a}, \bar{1}, v) = (2D(a) + \gamma a)v - (a\bar{1}, \bar{1}, v) \\
&\overset{(7)}{=} (2D(a) + \gamma a)v - (\bar{1} \cdot \bar{1}, a, v) - (\bar{1}, \bar{1}, v)a - (\bar{1}, a, v)\bar{1} \\
&= 2D(a)v + d(v)a + \gamma av,
\end{aligned}
$$

and similarly $\bar{v}\bar{a} = 2d(v)a + vD(a) + \gamma va$. Therefore, $V \cong B(V_{\bar{0}}, d, \gamma)$. $\qquad\square$

The converse is also true:

Proposition 6.2. *For any D-simple faithful associative and commutative bimodule (V, d) over (Γ, D) and any $\gamma \in \Gamma$ the bimodule $B(V, d, \gamma)$ is an irreducible faithful alternative superbimodule over the simple superalgebra $B = B(\Gamma, D, \gamma)$. Moreover, the B-superbimodules $B(V, d, \gamma)$ and $B(V', d', \gamma)$ are isomorphic if and only if the differential bimodules (V, d) and (V', d') over (Γ, D) are isomorphic.*

Proof. Let W be a nonzero subbimodule of V, then $W = W_{\bar{0}} + \bar{U}$ for some Γ-subbimodule U of $V_{\bar{0}}$. We show that $W_{\bar{0}} \neq 0$. Assume on the contrary that $W_{\bar{0}} = 0$. Then $\bar{v} \cdot \bar{1} = \bar{1} \cdot \bar{v} = 0$ for any $\bar{v} \in W_{\bar{1}} = \bar{U}$. Hence $d(v) = 0$ for any $v \in U$. It follows that U is a nonzero D-submodule of $V_{\bar{0}}$. Thus $U = V_{\bar{0}}$ and $0 = d(va) = vD(a)$ for any $v \in V_{\bar{0}}, a \in \Gamma$. Consequently, $D(\Gamma) = 0$, a contradiction.

Therefore $W_{\bar{0}}$ is a nonzero Γ-subbimodule of $V_{\bar{0}}$ and $\overline{W_{\bar{0}}} \subseteq W_{\bar{1}}$. For any $v \in W_{\bar{0}}$, $\bar{1} \cdot \bar{v} = \gamma v + d(v) \in W_{\bar{0}}$. Hence $d(W_{\bar{0}}) \subseteq W_{\bar{0}}$ and $W_{\bar{0}} = V_{\bar{0}}$.

Finally, if the superbimodules $B(V, d, \gamma)$ and $B(V', d', \gamma)$ are isomorphic then their even parts $V_{\bar{0}}$ and $V'_{\bar{0}}$ are isomorphic associative and commutative bimodules over Γ. Moreover, the derivations d and d' are defined via the same mapping $v \mapsto 2(v, \bar{1}, \bar{1})$, hence $(V, d) \cong (V', d')$. □

Clearly, for any D-simple algebra (Γ, D) the regular bimodule with derivation $\mathrm{Reg}\,(\Gamma, D)$ is a D-simple bimodule over (Γ, D). Moreover, for any $a \in \Gamma$ the map $D + R_a : \Gamma \to \Gamma$ is a derivation of the bimodule $\mathrm{Reg}\,\Gamma$. In fact, for any $v \in \mathrm{Reg}\,\Gamma$, $b \in \Gamma$ we have

$$(D + R_a)(vb) = D(vb) + vba = D(v)b + vD(b) + vba = (D + R_a)(v)b + vD(b).$$

It is clear that the differential bimodule $(\mathrm{Reg}\,\Gamma, D + R_a)$ is D-simple as well.

Proposition 6.3. *The D-bimodules $(\mathrm{Reg}\,\Gamma, D + R_a)$ and $(\mathrm{Reg}\,\Gamma, D + R_b)$ over (Γ, D) are isomorphic if and only if there exists an invertible $c \in \Gamma$ such that $D(c) = (a - b)c$. In particular, if $a, b \in F$ then $(\mathrm{Reg}\,\Gamma, D + R_a) \cong (\mathrm{Reg}\,\Gamma, D + R_b)$ if and only if $a - b \in Spec\,D$.*

Proof. Let $\varphi : (\mathrm{Reg}\,\Gamma, D + R_a) \to (\mathrm{Reg}\,\Gamma, D + R_b)$ be an isomorphism of D-bimodules over (Γ, D). Let $c = \varphi(1)$, then for any $x \in \Gamma$ we have $\varphi(x) = \varphi(1 \cdot x) = \varphi(1)x = cx$. Since φ is an isomorphism, $c\Gamma = \Gamma$ and c is invertible. Futhermore,

$$
\begin{aligned}
\varphi((D + R_a)(x)) &= cD(x) + cax = (D + R_b)(\varphi(x)) = D(cx) + bcx \\
&= D(c)x + cD(x) + bcx,
\end{aligned}
$$

which gives $(D(c) - (a - b)c)\Gamma = 0$. Since Γ is D-simple, this proves that $D(c) = (a - b)c$. Conversely, if c is an invertible element that satisfies this relation then the application $x \mapsto cx$ is an isomorphism of D-bimodules $(\mathrm{Reg}\,\Gamma, D + R_a)$ and $(\mathrm{Reg}\,\Gamma, D + R_b)$. It remains to notice that if $\lambda \in Spec\,D$, that is, $D(c) = \lambda c$ for some $c \in \Gamma$ then $c\Gamma$ is a D-ideal in Γ, hence $c\Gamma = \Gamma$ and c is invertible. □

We now describe the faithful D-simple bimodules over a finite dimensional D-simple associative commutative algebra (Γ, D) in the case when the ground field F is algebraically closed. Observe that due to [19] in this case $\Gamma \cong F[t_1, \ldots, t_n]/(t_1^3, \ldots, t_n^3)$.

Theorem 6.3. *Let (Γ, D) be a D-simple associative commutative algebra over an algebraically closed field F of characteristic 3, and let (V, d) be a D-simple*

finite dimensional associative and commutative bimodule over (Γ, D). Then there exist $\lambda \in F$ and $v \in V$ such that $V = v\Gamma$ and the action of the derivation d is given by $d(va) = v(\lambda a + D(a))$. In other words, the bimodule $(v\Gamma, d)$ is isomorphic to the D-bimodule $(\operatorname{Reg}\Gamma, D + \lambda)$. The parameter λ is defined uniquely modulo $\operatorname{Spec} D$.

Proof. Since the field F is algebraically closed and V is finite dimensional, the linear operator $d : V \to V$ has an eigen vector v with an eigen value $\lambda \in F$. For any $a \in \Gamma$ we have $d(va) = d(v)a + vD(a) = v(\lambda a + D(a))$, which proves that $v\Gamma$ is a d-subbimodule of V. Clearly, $v\Gamma \neq 0$ and therefore $V = v\Gamma$.

Consider the mapping $\varphi : \operatorname{Reg}\Gamma \to V$, $a \mapsto va$. Clearly, φ is an isomorphism of Γ-modules. Moreovere, we have

$$\varphi((D + \lambda)(a)) = v(D(a) + \lambda a) = d(va) = d(\varphi(a)),$$

which shows that $(V, d) \cong (\operatorname{Reg}\Gamma, D + \lambda)$. The last statement follows from Proposition 6.3. $\qquad \square$

As a result, for the finite dimensional alternative superalgebras over an algebraically closed field we have a complete description of irreducible nontrivial bimodules, which are not associative.

Theorem 6.4. *Let A be a finite dimensional alternative superalgebra over an algebraically closed field F, V be an irreducible faithful nontrivial alternative superbimodule over A, which is not associative. Then either $\dim A \leq 2$, $A_{\bar{1}} = Fx$, $x^2 = 0$ and $V \cong V^\varepsilon(1|1)$, $\varepsilon^2 + \varepsilon + 1 = 0$, or the superalgebra A is simple and V is either isomorphic to one of the bimodules $\operatorname{Reg} A$, $(\operatorname{Reg} A)^{\operatorname{op}}$, or one of the following cases holds:*

- *$A = F[\sqrt{1}]$ and V is one of the two-dimensional bimodules of types 1) – 3) described in Theorem 2.2;*

- *char $F = 3$, $A = B(1|2)$, $V = V_{\lambda, \mu}(3|3)$;*

- *char $F = 3$, $A = B(\Gamma, D, \gamma)$, $V = B(\operatorname{Reg}\Gamma, D + \lambda, \gamma)$, $\lambda \notin \operatorname{Spec} D$;*

- *char $F = 2$, $A = \mathrm{M}_{1|1}(F)$, $V = \operatorname{Cay} L$, L is a minimal $\mathbb{Z}_2$-graded left ideal of $\mathrm{M}_{1|1}(F)$;*

- *char $F = 2$, $A = \mathrm{M}_2(F)[\sqrt{1}]$, $V = (\operatorname{Cay} L)[\sqrt{1}]$, L is a minimal left ideal of $\mathrm{M}_2(F)$.*

Observe that there is a misprint in the announcement of this result in [22]: the case $A = B(\Gamma, D, \gamma)$ is omitted.

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