STRONGLY PRIME ALGEBRAIC LIE PI-ALGEBRAS

ANTONIO FERNÁNDEZ LÓPEZ

Dedicated to Professor Efim Zelmanov on the occasion of his 60th birthday

ABSTRACT. In a recent paper by the author and Artem Golubkov, it was proved that a strongly prime Lie PI-algebra with an algebraic adjoint representation over an algebraically closed field of characteristic 0 is simple and finite dimensional. In this note we derive this result from a more general one on strongly prime Lie PI-algebras with abelian minimal inner ideals, which is closely related to the intrinsic characterization of simple finitary Lie algebras with abelian minimal inner ideals.

INTRODUCTION

Theorem A. Let A be a prime associative algebra over a field $\mathbb{F}$. By using classical theory of associative PI-algebras (see Cohn's book [5]), it is easy to show that if A is algebraic and satisfies a polynomial identity, then A is simple and finite dimensional over its center, this being an algebraic extension of $\mathbb{F}$.

This can be reformulated as follows: Such an algebra A is simple, has *finite capacity* (A is unital and $1 = e_1 + \cdots + e_n$ is a sum of orthogonal *division idempotents*, i.e., e_iAe_i is a division algebra), and its center is an algebraic extension of $\mathbb{F}$.

Theorem J. Among the fundamental theorems proved by Zelmanov in [17], we find, as a small treasure, the following Jordan analog of Theorem A. Let J be a prime nondegenerate Jordan algebra over a field $\mathbb{F}$ of characteristic not 2. If J is algebraic and satisfies a polynomial identity which is not an s-identity, then J is simple, has *finite capacity* (J is unital and $1 = e_1 + \cdots + e_n$ is a sum of orthogonal *division idempotents*, i.e., $U_{e_i}J$ is a division Jordan algebra, uniquely determined up to isotopy), and its (associative) center is an algebraic extension of $\mathbb{F}$. Moreover, J is either finite dimensional over its center or the Jordan algebra of a nondegenerate symmetric bilinear form on an infinite dimensional vector space over an algebraic extension of $\mathbb{F}$.

Key words and phrases. Lie algebra, Jordan algebra, polynomial identity, algebraic adjoint representation, Engel and Jordan elements, finitary algebras.

Supported by the MEC and Fondos FEDER, MTM2014-52470-P.

Theorem L. Let L be a prime nondegenerate Lie algebra over a field $\mathbb{F}$ of characteristic 0. If L has an algebraic adjoint representation and satisfies a polynomial identity, then L is simple and finite dimensional over its centroid, this being an algebraic extension of $\mathbb{F}$. This theorem is proved in [8] by considering first the particular case (referred to as **Theorem F**) that $\mathbb{F}$ is algebraically closed (where the existence of a nonzero Engel element is assured, thus allowing the use of Jordan techniques as explained later), and then reducing the general case to this particular one via a tightened scalar extension and Zelmanov's theorem on local finiteness of Lie PI-algebras with an algebraic adjoint representation [18].

In this note, we derive Theorem F from a more general result on strongly prime Lie PI-algebras with abelian minimal inner ideals, which is closely related to the intrinsic characterization of simple finitary Lie algebras with abelian minimal inner ideals given in [6, Theorem 5.3]. In any strongly prime Lie PI-algebra with an algebraic adjoint representation over a field of characteristic 0, abelian minimal inner ideals occur as soon as we have nonzero Engel elements, a fact that is proved here by means of Theorem J and the Lie-Jordan connection [7]. The existence of a nonzero Engel element is automatic if the the algebra has a nontrivial finite grading, in particular, if the ground field is algebraically closed, but does not hold in general. For the sake of completeness, we will give an outline of the proof of Theorem J recalling the necessary definitions.

1. Common features of Lie and Jordan algebras

1. Throughout this note, and unless specified otherwise, we will be dealing with Lie algebras L ([10], [12]), with $[x, y]$ denoting the Lie product and ad_x the adjoint map determined by x over a field $\mathbb{F}$, and with *linear* Jordan algebras J ([13], [16]), with Jordan product $x \cdot y$, multiplication operators $m_x : y \mapsto x \cdot y$, quadratic operators $U_x = 2m_x^2 - m_{x^2}$ and triple product $\{x, y, z\} = U_{x+z}y - U_x y - U_z y$, over a field $\mathbb{F}$ of characteristic different from 2.

An associative algebra A (over a field of characteristic different from 2) gives rise to a Lie algebra A^- with Lie product $[x, y] := xy - yx$, and a linear Jordan algebra A^+ with Jordan product $x \cdot y := 1/2(xy + yx)$. A Jordan algebra J is said to be *special* if it is isomorphic to a subalgebra of A^+ for some associative algebra A.

2. An element $x \in J$ is called an *absolute zero divisor* if $U_x = 0$. We say J is *nondegenerate* if it has no nonzero absolute zero divisors, *semiprime* if $B^2 = 0$ implies $B = 0$, and *prime* if $B \cdot C = 0$ implies $B = 0$ or $C = 0$, for any ideals B, C of J.

Similarly, given a Lie algebra L, $x \in L$ is an *absolute zero divisor* of L if $\mathrm{ad}_x^2 = 0$ (for Lie algebras over a field of characteristic 2, standard definition of absolute zero divisor or cover of a thin sandwich requires $\mathrm{ad}_x^2 = 0 = \mathrm{ad}_x \mathrm{ad}_y \mathrm{ad}_x$, $y \in L$); L is *nondegenerate* if it has no nonzero absolute zero divisors, *semiprime* if $[B, B] = 0$ implies $B = 0$, and *prime* if $[B, C] = 0$ implies $B = 0$ or $C = 0$, for any ideals B, C of L. Nondegeneracy for both Jordan and Lie algebras implies semiprimeness, but the converse does not hold. A Jordan or Lie algebra is said to be *strongly prime* if it is prime and nondegenerate. *Simplicity,* for both Jordan and Lie algebras, means nonzero product and the absence of nonzero proper ideals.

3. Inner Ideals. An *inner ideal* of a Jordan algebra J is a vector subspace B of J such that $\{B, J, B\} \subseteq B$. Similarly, an *inner ideal* of a Lie algebra L is a vector subspace B of L such that $[[B, L], B] \subseteq B$. An *abelian inner ideal* of L is an inner ideal B which is also an abelian subalgebra, i.e. $[B, B] = 0$.

For any element $a \in J$, $U_a J$ is an inner ideal of J, as follows from the Fundamental Jordan Identity $U_{U_x y} = U_x U_y U_x$, $x, y \in J$. Hence a nonzero subspace B of a nondegenerate algebra J is a minimal inner ideal if and only if $B = U_b J$ for any nonzero $b \in J$. As will be seen in Section 4, only a special kind of elements in Lie algebras yield inner ideals in a similar way.

4. Centers. For a Lie algebra L, the *center* of L, denoted by $Z(L)$, is the set of all $z \in L$ such that $[z, x] = 0$ for all $x \in L$. For a Jordan algebra J, the *center* of J is simply the *nucleus:*

$$Z(J) = \{z \in J : (z, x, y) = (x, z, y) = (x, y, z) = 0,\ x, y \in J\},$$

with $(a, b, c) = (a \cdot b) \cdot c - a \cdot (b \cdot c)$ *for all* $a, b, c \in J$.

2. Theorem J

Throughout this section J will denote a Jordan algebra over a field $\mathbb{F}$ of characteristic different from 2.

5. Division Jordan algebras. Let J be a Jordan algebra with 1. An element $x \in J$ is called *invertible* if there exists $y \in J$ such that $x \cdot y = 1$ and $x^2 \cdot y = x$. In this case U_x is invertible and the *inverse* of x, denoted by x^{-1} is uniquely determined: $x^{-1} = U_x^{-1} x$ [16, II.6.1.1-7]. A unital Jordan algebra in which every nonzero element is invertible is called a *division Jordan algebra.* If $J = A^+$ for an associative algebra A, then A^+ is a division Jordan algebra if and only if A is a division associative algebra [16, II.6.1.5].

6. Division idempotents. An idempotent e of a Jordan algebra J is called a *division idempotent* if the principal inner ideal U_eJ is minimal, equivalently, U_eJ is a division Jordan algebra with e as unit [16, I.5.1]. We note that if J is strongly prime, then the division Jordan algebra determined by a division idempotent is uniquely determined up to *isotopy*. (See [16, II.7.2] for definition.)

7. Capacity. A Jordan algebra J is said to have *capacity* n if J is unital and 1 can be written as a sum of n orthogonal division idempotents. By Jacobson's capacity theorem [16, I.5.2], any nondegenerate Jordan algebra having finite capacity is a direct sum of ideals each of which is a simple Jordan algebra of finite capacity.

8. Semiprimitive Jordan algebras. A Jordan algebra is said to be *semiprimitive* if has no quasi-invertible ideals (see [16, III.1.3.1] for definition), i.e., its Jacobson radical vanishes. Any semiprimitive Jordan algebra is nondegenerate [16, III.1.6.1].

9. I-algebras. A Jordan algebra J over a field $\mathbb{F}$ is said to be *algebraic* if every element $x \in J$ is a root of a nontrivial polynomial in $\mathbb{F}[\xi]$. As proved in [16, I.8.1 (Algebraic I Proposition)], any algebraic Jordan algebra J is an *I-algebra*, i.e., every non-nil inner ideal of J contains a nonzero idempotent.

Theorem 2.1. (McCrimmon) *Any semiprimitive I-algebra having no infinite family of nonzero orthogonal idempotents is unital and has finite capacity.*

Proof. See [16, I.8.1 (I-Finite Capacity Theorem)] for a sketch of the proof. □

10. Jordan PI-algebras. A Jordan polynomial $p(x_1, \ldots, x_n)$ of the free Jordan $\mathbb{F}$-algebra $J(X)$ is said to be an *s-identity* if it vanishes in all special Jordan algebras, but not in all Jordan algebras. A Jordan algebra J satisfying a polynomial identity which is not an s-identity is called a *Jordan PI-algebra*.

Lemma 2.2. (Zelmanov) *Let J be a strongly prime Jordan PI-algebra. Then J has no infinite family of nonzero orthogonal idempotents.*

Proof. Let $e \in J$ be a nonzero idempotent. It follows from [1, Corollary 3.3] that the unital Jordan algebra U_eJ inherits strong primeness of J. Moreover, if U_eJ is strictly contained in J, then U_eJ is special by [17, Lemma 20]. This reduces the proof of the lemma to the case that J is special, which is proved in [17, Lemma 19]. □

Theorem 2.3. (Zelmanov) *Let J be a strongly prime algebraic Jordan PI-algebra over the field $\mathbb{F}$. Then J is simple and has finite capacity. Moreover, its center is an algebraic field extension of $\mathbb{F}$.*

Proof. Since J is nondegenerate and PI, it follows from Zelmanov PI-Radical Theorem [17, Theorem 4] that J does not contain nonzero nil ideals. This and the fact that J is an I-algebra (9) imply that J is semiprimitive (otherwise the Jacobson radical of J would contain a nonzero idempotent, a contradiction). Moreover, we have by Lemma 2.2 that J has no infinite family of nonzero orthogonal idempotents. Therefore J has finite capacity by Theorem 2.1. Since J is prime, it is actually simple by Jacobson's capacity theorem, and its center is an algebraic extension of $\mathbb{F}$. □

Remark 2.4. Zelmanov actually proves in [17, Lemma 25] that a prime semiprimitive algebraic Jordan PI-algebra is locally finite over its centroid. He first shows that such a Jordan algebra has finite capacity and then applies Jacobson's capacity theorem to reduce the proof to each of the four kinds of simple Jordan algebras of finite capacity.

3. Theorem A

Let $J = A^+$ for an associative algebra A. By the elemental characterization of semiprimeness of associative algebras ($aAa = 0 \Rightarrow a = 0$), A is semiprime if and only if A^+ is nondegenerate, and by [9, Theorem 1.1], A is prime (resp. simple) if and only if A^+ is strongly prime (resp. simple). It is clear that A is unital if and only if A^+ is unital, A and A^+ share the same idempotents, and the relation of orthogonality for idempotents is the same in A as in A^+. Moreover, for any idempotent $e \in A$, $(eAe)^+ = U_e A^+$ and, by (5), eAe is a division associative algebra if and only if $(eAe)^+$ is a division Jordan algebra. These facts altogether prove that **Theorem J** is a Jordan extension of **Theorem A**.

4. The Lie-Jordan connection

Throughout this section L will denote a Lie algebra over a field $\mathbb{F}$ of characteristic different from 2 and 3.

11. Engel and Jordan Elements. An element $a \in L$ is called *Engel* if ad_a is a nilpotent operator. In this case, the nilpotence index of ad_a is called the *index* of a. Engel elements of index at most 3 are called *Jordan elements.* It is easy to verify that any element a of an associative algebra A such that $a^2 = 0$ is a Jordan element of the Lie algebra A^-.

Clearly, any element of an abelian inner ideal is a Jordan element. Conversely, by [4, Lemma 1.8], any Jordan element a generates the *principal* abelian inner ideal $\mathrm{ad}_a^2 L$.

As in the Jordan case, this result follows from an analog of the Fundamental Jordan Identity:

$$\mathrm{ad}^2_{\mathrm{ad}^2_a x} = \mathrm{ad}^2_a \mathrm{ad}^2_x \mathrm{ad}^2_a$$

which holds for any Jordan element a and any $x \in L$ [4, Lemma 1.7(iii)]. This identity is a good justification for the use of term Jordan element. Another reason for adopting this terminology is the following:

12. Jordan Algebra at a Jordan Element. Let a be a Jordan element of a Lie algebra L. It was proved in [7, Theorem 2.4] that the underlying vector space L with the new product defined by $x \cdot_a y := [[x, a], y]$ is a nonassociative algebra, denoted by $L^{(a)}$, such that

(i) $\mathrm{Ker}_L a := \{x \in L : [a, [a, x]] = 0\}$ is an ideal of $L^{(a)}$.
(ii) $L_a := L^{(a)}/\mathrm{Ker}_L a$ is a Jordan algebra, called *the Jordan algebra of* L at a.

We denote by $x \mapsto \bar{x}$ the natural epimorphism of $L^{(a)}$ onto L_a and by $U^{(a)}_{\bar{x}}$ the U-operator of $\bar{x}$ in L_a. As proved in [8], many properties of a Lie algebra can be transferred to its Jordan algebras. Moreover, the nature of the Jordan element in question is reflected in the structure of the attached Jordan algebra. These facts turn out to be crucial for applications of Jordan theory to Lie algebras.

Lemma 4.1. *Suppose that L is nondegenerate and let $0 \neq a \in L$ be a Jordan element. Then the following conditions are equivalent:*

(i) $\mathrm{ad}^2_a L$ *is an abelian minimal inner ideal.*
(ii) L_a *is a division Jordan algebra.*

In this case, $a \in \mathrm{ad}^2_a L$.

Proof. We know by [7, 2.15(i)] that for any Jordan element $a \in L$, the Jordan algebra L_a inherits nondegeneracy from L, and by [7, 2.14] there is a one-to-one order-preserving correspondence between the principal inner ideals of L contained in $\mathrm{ad}^2_a L$ and the principal inner ideals of L_a. Hence $\mathrm{ad}^2_a L$ is minimal if and only if L_a does not contain proper inner ideals; but it is the absence of inner ideals what characterizes the division Jordan algebras [16, II.18.1.4]. Now it follows from [7, 2.15(ii)] that just assuming that L_a is unital and L nondegenerate, there exists $b \in L$ such that $a = [[a, b]a]$. This completes the proof of the lemma. □

13. Division elements. Let L be a nondegenerate Lie algebra. A nonzero Jordan element $a \in L$ will be called a *Jordan element* if it satisfies the equivalent conditions of Lemma 4.1.

Lemma 4.2. *Let $0 \neq a \in L$ be a Jordan element of a nondegenerate Lie algebra L and let $e \in L$ be such that $\bar{e}$ is a division idempotent of L_a. Then $\mathrm{ad}_a^2 e$ is a division element of L.*

Proof. $U_{\bar{e}}^{(a)} L_a$ is a minimal inner ideal of L_a. Hence, by [7, 2.14], $\mathrm{ad}^2_{\mathrm{ad}_a^2 e} L$ is an abelian minimal inner ideal of L, equivalently, $\mathrm{ad}_a^2 e$ is a division element of L, as required. □

5. Theorem F

Throughout this section L will denote a Lie algebra over a field $\mathbb{F}$ of characteristic 0.

14. Simple Lie algebras with abelian minimal inner ideals. Let L be a simple Lie algebra containing abelian minimal inner ideals. It was proved in [6, 2.2], although in a notation different from that used here, that the division Jordan algebra L_a defined by a division element $a \in L$ is independent of the choice of the element up to *isotopy*, equivalently, the Jordan pair (L_a, L_a) is an invariant of the Lie algebra L up to isomorphism of Jordan pairs. (See [15, 1.12] for definitions.)

15. Finitary algebras. Following [3], a Lie algebra L is said to be *finitary* (over $\mathbb{F}$) if it is isomorphic to a subalgebra of $\mathcal{F}(X)^-$, the Lie algebra consisting of all finite rank operators on a vector space X over $\mathbb{F}$.

Recall that an (associative, Jordan or Lie) algebra A is said to be *locally finite* if every finitely generated subalgebra of A is finite dimensional.

Lemma 5.1. *Finitary Lie algebras L are locally finite.*

Proof. Let $L \leq \mathcal{F}(X)^-$. It is enough to see that the associative algebra $\mathcal{F}(X)$ is locally finite, but this follows from Litoff's theorem [11, IV.15.Theorem 3]. □

Finitary Lie algebras do not necessarily contain abelian minimal inner ideals. For instance, the finitary orthogonal algebra defined by a vector space with an anisotropic symmetric bilinear form over a field $\mathbb{F}$ does not contain any nonzero abelian inner ideal.

A simple Lie algebra over a field $\mathbb{F}$ is said to be *central* if its centroid is one-dimensional over $\mathbb{F}$. It was proved in [6, Theorem 5.3]:

Proposition 5.2. *A central simple Lie algebra containing abelian minimal inner ideals is finitary if and only if its associated division Jordan algebra is PI.*

Remark 5.3. Using Zelmanov's classification of division Jordan PI-algebras (see [16, I.7.3, I.7.4]) and computing their isotopes in each one of the different four types [16,

II.7.3.1, II.7.4.1, II.7.5.1, II.7.5.2], one checks that the isotope of a division Jordan PI-algebra is again a division Jordan PI-algebra. This proves that the characterization of central simple finitary Lie algebras with abelian minimal inner ideals given in the preceding proposition makes sense.

16. Lie PI-algebras. A Lie algebra satisfying a nontrivial polynomial identity is called a *Lie PI-algebra.*

Lemma 5.4. *Let L be a strongly prime Lie PI-algebra over a field of characteristic* 0 *containing abelian minimal inner ideals. Then L is simple and finite dimensional over its centroid.*

Proof. Let B be an abelian minimal inner ideal of L and denote by S the ideal of L generated by B. By [6, 1.14, 1-15], B is an (abelian) minimal inner ideal of S and S is a simple Lie algebra. Let $0 \neq b \in B$. It follows from Lemma 4.1 and [8, Proposition 4.2(iv)] that S_b is a division Jordan PI-algebra. Hence, by Proposition 5.2, S is finitary over its centroid and therefore locally finite by Lemma 5.1. Since S is also PI, S is actually finite dimensional by [2, Theorem 2]. By primeness, L can be embedded in $\mathrm{Der}(S)$ via the adjoint representation. Since any derivation on S is inner, $L = S$ is simple and finite dimensional over its centroid. □

Remark 5.5. Although the proof of the preceding lemma can seem to be elemental, it is ultimately based on the structure theorem for simple Lie algebras with a finite grading, given by Zelmanov in [19], which is the key tool to prove the intrinsic characterization of simple finitary Lie algebras with abelian minimal inner ideals.

17. Algebraic Lie algebras. A Lie algebra L over a field $\mathbb{F}$ is said to be *algebraic* if for each x in L the inner derivation ad_x is a root of a nonzero polynomial in $\mathbb{F}[\xi]$.

Lemma 5.6. *Let L be a strongly prime algebraic Lie algebra over a field $\mathbb{F}$ of characteristic* 0 *containing abelian minimal inner ideals. Then the centroid of L is an algebraic field extension of $\mathbb{F}$.*

Proof. Let $a \in L$ be a division element and let $b \in L$ be such that $a = [[a, b], a]$ (Lemma 4.1). By [8, Proposition 4.2(iii)], the division Jordan algebra L_a is algebraic over $\mathbb{F}$. Denote by $\Gamma(L)$ the centroid of L and by $Z(L_a)$ the center of L_a. By primeness of L, $\Gamma(L)$ is an integral domain and L_a can be regarded as an algebra over $\Gamma(L)$. We claim that the map $f : \Gamma(L) \to Z(L_a)$ given by $\gamma \mapsto \gamma\bar{b} = \overline{\gamma(b)}$, $\gamma \in \Gamma(L)$, is a monomorphism of $\mathbb{F}$-algebras. Linearity of f is clear and $\overline{\gamma(b)} = 0 \Leftrightarrow [[a, \gamma(b)], a] = \gamma([[a, b], a]) = \gamma(a) = 0$, which implies $\gamma = 0$ by primeness of L, so f is injective. Finally, for $\eta, \gamma \in \Gamma(L)$, we have

$$\overline{\eta\gamma(b)} = \overline{\eta\gamma([[b,a],b]} = \overline{[[\eta(b),a],\gamma(b)]} = \eta\bar{b} \cdot_a \gamma\bar{b},$$

which proves that f is an algebra homomorphism. Since $Z(L_a)$ is an algebraic field extension of $\mathbb{F}$, $\Gamma(L) \leq Z(L_a)$ is an algebraic extension of $\mathbb{F}$; but $\Gamma(L)$ is a domain, so $\Gamma(L)$ is an algebraic *field* extension, as required. □

Proposition 5.7. *Let L be a strongly prime algebraic Lie PI-algebra over a field $\mathbb{F}$ of characteristic 0 containing a nonzero Engel element. Then:*

(i) *L contains abelian minimal inner ideals.*

(ii) *The centroid of L is an algebraic field extension of $\mathbb{F}$ and L is simple and finite dimensional over its centroid.*

Proof. (i) Let $a \in L$ be a nonzero Jordan element, whose existence is guaranteed by Kostrikin's Descendent Lemma [14, Lemma 2.1.1]. By [8, Proposition 4.2], L_a is a strongly prime algebraic Jordan PI-algebra over $\mathbb{F}$, and hence, by Theorem 2.3, L_a is simple and has finite capacity. In particular, L_a contains a division idempotent, but, as proved in Lemma 4.2, any division idempotent of L_a yields a division element in L and therefore an abelian minimal inner ideal, as required.

(ii) It follows from (i) together with Lemmas 5.4 and 5.6. □

18. Extremal Elements. Let L be a Lie algebra over a field $\mathbb{F}$. A nonzero element $a \in L$ is said to be *extremal* if $\mathrm{ad}_a^2 L = \mathbb{F}a$, i.e., a generates a one-dimensional inner ideal. As proved in [6, Lemma 5.4]:

Proposition 5.8. *A simple Lie algebra over a field $\mathbb{F}$ containing an extremal element is central.*

Proposition 5.9. *Let L be a strongly prime Lie PI-algebra over a field $\mathbb{F}$ of characteristic 0 containing an extremal element. Then L is simple and finite dimensional over $\mathbb{F}$.*

Proof. Let $a \in L$ be an extremal element. By [6, 1-15], the ideal S generated by a is simple as an algebra, and locally finite by [18, Lemma 15]. Since S is PI, S is actually finite dimensional over its centroid [2, Theorem 2]. As in the proof of Lemma 5.4, we conclude that L is simple and finite dimensional over $\mathbb{F}$, since L is central as quoted above. □

Corollary 5.10. [8, Proposition 5.1] *Let L be a strongly prime algebraic Lie PI-algebra over an algebraically field $\mathbb{F}$ of characteristic 0. Then L is simple and finite dimensional over $\mathbb{F}$.*

Proof. Since L is algebraic, it follows from [8, Corollary 2.3] that L contains a nonzero Engel element. Hence L contains a division element by Proposition 5.7(i). But by [8, Proposition 4.3], under the condition that $\mathbb{F}$ is algebraically closed, any division element in L is extremal. Now Proposition 5.9 applies to finish the proof. □

Acknowledgement

The author wishes to thank Esperanza Sánchez and Miguel Cabrera for the careful reading of the manuscript and their suggestions to improve the presentation of the paper.

References

[1] J. A. Anquela, T. Cortés, O. Loos, and K. McCrimmon, An elemental characterization of strong primeness in Jordan systems, *J. Pure Appl. Algebra* **109** (1999), 23-36.
[2] Y. A. Bakhturin, Simple Lie algebras satisfying a nontrivial identity, *Serdica* **2** (1976), 241-246.
[3] A. A. Baranov, Finitary simple Lie algebras, *J. Algebra* **219** (1999), 299-329.
[4] G. Benkart, On inner ideals and ad-nilpotent elements of Lie algebras, *Trans. Amer. Math. Soc.* **232** (1977), 61-81.
[5] P. M. Cohn, *Further Algebra and Applications*, Springer, 2003.
[6] C. Draper, A. Fernández López, E. García, and M. Gómez Lozano, The socle of a non-degenerate Lie algebra, *J. Algebra* **319** (2008), 2372-2394.
[7] A. Fernández López, E. García, and M. Gómez Lozano, The Jordan algebras of a Lie algebra, *J. Algebra* **308** (2007), 164-177.
[8] A. Fernández López and A. Yu. Golubkov, Lie algebras with an algebraic adjoint representation revisited, *Manuscripta Math.* **140** (2013), 363-376.
[9] I. N. Herstein, *Topics in Ring Theory*, U. of Chicago Lectures Notes, 1965.
[10] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Grad. Texts in Math. 9, Springer-Verlag, Heidelberg-Berlin- New York, 1972.
[11] N. Jacobson, *Structure of Rings*, American Mathematical Society, 1956.
[12] N. Jacobson, *Lie Algebras*, Interscience Publishers, New York, 1962.
[13] N. Jacobson, *Structure and Representations of Jordan Algebras*, American Mathematical Society, 1968.
[14] A. I. Kostrikin, *Around Burnside.* A Series of Modern Surveys in Mathematics. Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, Hong Kong, 1990.
[15] O. Loos, *Jordan Pairs*, Lecture Notes in Math, vol. 460, Springer-Verlag, Heidelberg-Berlin- New York, 1975.

[16] K. McCrimmon, *A Taste on Jordan Algebras*, Springer, 2004.
[17] E. I. Zelmanov, Absolute zero-divisors and algebraic Jordan algebras, *Siberian Math. J.* **23** (1982), 100-116.
[18] E. I. Zelmanov, Lie algebras with an algebraic adjoint representation *Math. USSR Sb.* **49** (1984), 537-552.
[19] E. I. Zelmanov, Lie algebras with a finite grading *Math. USSR Sb.* **52** (1985), 347-385.

Departamento de Álgebra, Geometría y Topología, Universidad de Málaga, Spain
E-mail address: emalfer@uma.es