Manifolds of algebraic elements in JB*-triples

José M. Isidro *
Facultad de Matemáticas,
Universidad de Santiago,
Santiago de Compostela, Spain.
jmisidro@zmat.usc.es

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Abstract

Given a complex Hilbert space H, we study the differential geometry of the manifold $\mathcal{A}$ of normal algebraic elements in $Z = \mathcal{L}(H)$. We represent $\mathcal{A}$ as a disjoint union of connected subsets $\mathcal{M}$ of Z. Using the algebraic structure of Z, a torsionfree affine connection ∇ (that is invariant under the group $\mathsf{Aut}\,(Z)$ of automorphisms of Z) is defined on each of these connected components and the geodesics are computed. In case $\mathcal{M}$ consists of elements that have a fixed finite rank r, $(0 < r < \infty)$, $\mathsf{Aut}\,(Z)$-invariant Riemann and Kähler structures are defined on $\mathcal{M}$ which in this way becomes a totally geodesic symmetric holomorphic manifold. Similar results are established for the manifold of algebraic elements in an abstract JB*-triple.

Keywords. JB*-triples, Grassmann manifolds, Riemann manifolds.

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1 Introduction

In this paper we are concerned with the differential geometry of some infinite-dimensional Grassmann manifolds in $Z\colon = \mathcal{L}(H)$, the space of bounded linear operators $z\colon H \to H$ in a complex Hilbert space H. Grassmann manifolds are a classical object in Differential Geometry and in recent years several authors have considered them in the Banach space setting. Besides the Grassmann structure, a Riemann and a Kähler structure has sometimes been defined even in the infinite-dimensional setting. Let us recall some aspects of the topic that are relevant for our purpose.

The study of the manifold of minimal projections in a finite-dimensional simple formally real Jordan algebra was made by U. Hirzebruch in [6], who proved that such a manifold is a compact symmetric Riemann space of rank 1, and that every such a space arises in this way. Later on, Nomura in [18, 19] established similar results for the manifold of fixed finite rank projections in a topologically simple real Jordan-Hilbert algebra. In [8], the authors studied the Riemann and Kähler structure of the manifold of finite rank projections in Z without the use of any global scalar product. As pointed out there, the Jordan-Banach structure of Z encodes information about the differential geometry of some manifolds naturally associated to it, one of which is the manifold of algebraic elements in Z. On the other hand, the Grassmann manifold of all projections in Z has been discussed by Kaup in [10] and [13]. See also [1, 7] for related results.

It is therefore reasonable to ask whether a Riemann structure can be defined in the set of algebraic elements in Z, and how does it behave when it exists. We restrict our considerations to the set $\mathcal{A}$ of all normal algebraic elements in Z that have finite rank. Remark that the assumption concerning the finiteness of the rank can not be dropped, as proved in [8]. Normality allows us to use spectral theory which is an essential tool. In the case $H = \mathbb{C}^n$, all elements in Z are algebraic (as any square matrix is a root of its characteristic polynomial) and have finite rank. Under the above restrictions $\mathcal{A}$ is represented as a disjoint union of connected subsets M of Z, each of which is

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invariant under $\mathsf{Aut}\,(Z)$ (the group of all C*-automorphisms of Z). Using algebraic tools, a holomorphic manifold structure and an $\mathsf{Aut}\,(Z)$-invariant affine connection ∇ are introduced on M and its geodesics are calculated. One of the novelties is that we take JB*-triple system approach instead of the Jordan-algebra approach of [18, 19]. As noted in [1] and [7], within this context the algebraic structure of JB*-triple acts as a substitute for the Jordan algebra structure. In case M consists of elements that have a fixed finite rank r, $(0 < r < \infty)$, the JB*-triple structure provides a *local scalar product* known as the *algebraic metric* of Harris ([2], prop. 9.12). Although Z is not a Hilbert space, the use of the algebraic scalar product allows us to define an $\mathsf{Aut}\,(Z)$-invariant Riemann and a Kähler structure on M. We prove that ∇ is the Levi-Civita and the Kähler connection of M, and that M is a symmetric holomorphic manifold on which $\mathsf{Aut}^\circ(Z)$ acts transitively as a group of isometries.

The role that projections play in the study of the algebra $Z = \mathcal{L}(H)$ is taken by tripotents in the study of a JB*-triple system. A spectral calculus and a notion of algebraic element is available in the stetting of JB*-triples, and the manifold of all finite rank algebraic elements in a JB*-triple Z is studied in the final section.

2 Algebraic preliminaries.

For a complex Banach space X denote by $X_{\mathbb{R}}$ the underlying real Banach space, and let $\mathcal{L}(X)$ and $\mathcal{L}_{\mathbb{R}}(X)$ respectively be the Banach algebra of all bounded complex-linear operators on X and the Banach algebra of all bounded real-linear operators on $X_{\mathbb{R}}$. A complex Banach space Z with a continuous mapping $(a, b, c) \mapsto \{abc\}$ from $Z \times Z \times Z$ to Z is called a *JB*-triple* if the following conditions are satisfied for all $a, b, c, d \in Z$, where the operator $a\Box b \in \mathcal{L}(Z)$ is defined by $z \mapsto \{abz\}$ and $[\ ,\ ]$ is the commutator product:

1. $\{abc\}$ is symmetric complex linear in a, c and conjugate linear in b.
2. $[a\Box b,\, c\Box d] = \{abc\}\Box d - c\Box\{dab\}$.
3. $a\Box a$ is hermitian and has spectrum ≥ 0.
4. $\|\{aaa\}\| = \|a\|^3$.

If a complex vector space Z admits a JB*-triple structure, then the norm and the triple product determine each other. For $, x, y, z \in Z$ we write $L(x, y)(z) = (x\Box y)(z)$ and $Q(x, y)(z) \colon = \{xzy\}$. Note that $L(x, y) \in \mathcal{L}(Z)$ whereas $Q(x, y) \in \mathcal{L}_{\mathbb{R}}(Z)$, and that the operators $L_a = L(a, a)$ and $Q_a = Q(a, a)$ commute. A *derivation* of a JB*-triple Z is an element $\delta \in \mathcal{L}(Z)$ such that $\delta\{zzz\} = \{(\delta z)zz\} + \{z(\delta z)z\} + \{zz(\delta z)\}$ and an *automorphism* is a bijection $\phi \in \mathcal{L}(Z)$ such that $\phi\{zzz\} = \{(\phi z)(\phi z)(\phi z)\}$ for $z \in Z$. The latter occurs if and only if ϕ is a surjective linear isometry of Z. The group $\mathsf{Aut}\,(Z)$ of automorphisms of Z is a real Banach-Lie group whose Banach-Lie algebra is the set $\mathsf{Der}\,(Z)$ of all derivations of Z. The connected component of the identity in $\mathsf{Aut}\,(Z)$ is denoted by $\mathsf{Aut}^\circ(Z)$. Two elements $x, y \in Z$ are *orthogonal* if $x\Box y = 0$ and $e \in Z$ is called a *tripotent* if $\{eee\} = e$, the set of which is denoted by $\mathsf{Tri}\,(Z)$. For $e \in \mathsf{Tri}\,(Z)$, the set of eigenvalues of $e\Box e \in \mathcal{L}(Z)$ is contained in $\{0,\, \frac{1}{2},\, 1\}$ and the topological direct sum decomposition, called the *Peirce decomposition* of Z,

$$Z = Z_1(e) \oplus Z_{1/2}(e) \oplus Z_0(e). \tag{1}$$

holds. Here $Z_k(e)$ is the k- eigenspace of $e\Box e$ and the *Peirce projections* are

$$P_1(e) = Q^2(e), \qquad P_{1/2}(e) = 2(e\Box e - Q^2(e)), \qquad P_0(e) = \mathrm{Id} - 2e\Box e + Q^2(e).$$

We will use the *Peirce rules* $\{Z_i(e)\, Z_j(e)\, Z_k(e)\} \subset Z_{i-j+k}(e)$ where $Z_l(e) = \{0\}$ for $l \neq 0, 1/2, 1$. In particular, every Peirce space is a JB*-subtriple of Z and $Z_1(e)\Box Z_0(e) = \{0\}$. We note that $Z_1(e)$ is a complex unital JB*-algebra in the product $a \circ b \colon = \{aeb\}$ and involution $a^{\#} \colon = \{eae\}$. Let

$$A(e) \colon = \{z \in Z_1(e) \,:\, z^{\#} = z\}.$$

Then we have $Z_1(e) = A(e) \oplus iA(e)$. A tripotent e in a JB*-triple Z is said to be *minimal* if $e \neq 0$ and $P_1(e)Z = \mathbb{C}\, e$, and we let $\mathsf{Min}\,(Z)$ be the set of them. If $e \in \mathsf{Min}\,(Z)$ then $\|e\| = 1$. A JB*-triple Z may have no non-zero tripotents.

Let $\mathbf{e} = (e_1, \cdots, e_n)$ be a finite sequence of non-zero mutually orthogonal tripotents $e_j \in Z$, and define for all integers $0 \le j, k \le n$ the linear subspaces

$$\begin{aligned} Z_{j,j}(\mathbf{e}) &= Z_1(e_j) && 1 \le j \le n, \\ Z_{j,k}(\mathbf{e}) = Z_{k,j}(\mathbf{e}) &= Z_{1/2}(e_j) \cap Z_{1/2}(e_k) && 1 \le j,k \le n, \quad j \neq k, \\ Z_{0,j}(\mathbf{e}) = Z_{j,0}(\mathbf{e}) &= Z_1(e_j) \cap \bigcap_{k \neq j} Z_0(e_k) && 1 \le j \le n, \\ Z_{0,0}(\mathbf{e}) &= \bigcap_j Z_0(e_j). \end{aligned} \tag{2}$$

Then the following topologically direct sum decomposition, called the Peirce decomposition relative to $\mathbf{e}$, holds

$$Z = \bigoplus_{0 \le j \le k \le n} Z_{j,k}(\mathbf{e}). \tag{3}$$

The Peirce spaces multiply according to the rules $\{Z_{j,m} Z_{m,n} Z_{n,k}\} \subset Z_{j,k}$, and all products that cannot be brought to this form (after reflecting pairs of indices if necessary) vanish. In terms of this decomposition, the Peirce spaces of the tripotent $e := e_1 + \cdots + e_n$ are

$$\begin{aligned} Z_1(e) &= \bigoplus_{j,k} Z_{j,k}(\mathbf{e}) = \Big(\bigoplus_{1 \le j \le n} Z_{j,j}(\mathbf{e})\Big) \oplus \Big(\bigoplus_{\substack{1 \le j,k \le n \\ j \neq k}} Z_{j,k}(\mathbf{e})\Big), \\ Z_{1/2}(e) &= \bigoplus_{1 \le j \le n} Z_{0,j}(\mathbf{e}), \qquad Z_0(\mathbf{e}) = Z_{0,0}(\mathbf{e}). \end{aligned} \tag{4}$$

Recall that every C*-algebra Z is a JB*-triple with respect to the triple product $2\{abc\} := (ab^*c + cb^*a)$. In that case, every projection in Z is a tripotent and more generally the tripotents are precisely the partial isometries in Z. C*-algebra derivations and C*-automorphisms are derivations and automorphisms of Z as a JB*-triple though the converse is not true.

We refer to [11], [13], [16], [20] and the references therein for the background of JB*-triples theory.

3 Manifolds of algebraic elements in $\mathcal{L}(H)$.

From now on, Z will denote the C*-algebra $\mathcal{L}(H)$. An element $a \in Z$ is said to be *algebraic* if it satisfies the equation $p(a) = 0$ for some non identically null polynomial $p \in \mathbb{C}[X]$. By elementary spectral theory $\sigma(a)$, the spectrum of a in Z, is a finite set whose elements are roots of the algebraic equation $p(\lambda) = 0$. In case a is *normal* we have

$$a = \sum_{\lambda \in \sigma(a)} \lambda\, e_\lambda \tag{5}$$

where λ and e_λ are, respectively, the spectral values and the corresponding spectral projections of a. If $0 \in \sigma(a)$ then e_0, the projection onto $\ker(a)$, satisfies $e_0 \neq 0$ but in (5) the summand $0\, e_0$ is null and will be omitted. In particular, in (5) the numbers λ are non-zero pairwise distinct complex numbers and the e_λ are pairwise orthogonal non-zero projections. We say that a has *finite rank* if $\dim a(H) < \infty$, which always occurs if $\dim(H) < \infty$. Set $r_\lambda :=$ rank (e_λ). Then a has finite rank if and only if $r_\lambda < \infty$ for all $\lambda \in \sigma(a)\backslash\{0\}$ (the case $0 \in \sigma(a)$ and $\dim \ker a = \infty$ may occur).

Thus, every finite rank normal algebraic element $a \in Z$ gives rise to: (i) a positive integer n which is the cardinal of $\sigma(a)\backslash\{0\}$, (ii) an ordered n-uple $(\lambda_1, \cdots, \lambda_n)$ of numbers in $\mathbb{C}\backslash\{0\}$ which is the set of the pairwise distinct non-zero spectral values of a, (iii) an ordered n-uple $(e_1, \cdots, e_n)$ of non-zero pairwise orthogonal projections, and (iii) an ordered n-uple $(r_1, \cdots, r_n)$ where $r_k \in \mathbb{N}\backslash\{0\}$.

The spectral resolution of a is unique except for the order of the summands in (5), therefore these three n-uples are uniquely determined up to a permutation of the indices $(1, \cdots, n)$. The operator a can be recovered from the set of the first two ordered n-uples, a being given by (5).

Given the n-uples $\Lambda\colon=(\lambda_1,\cdots,\lambda_n)$ and $R\colon=(r_1,\cdots,r_n)$ in the above conditions, we let

$$M(n,\,\Lambda,\,R)\colon=\{\sum_k \lambda_k e_k\ :\ \ e_j e_k=0\ \text{ for } j\neq k,\ \text{rank }(e_k)=r_k,\ 1\le j,\,k\le n\,\} \tag{6}$$

be the set of the elements (5) where the coefficients λ_k and ranks r_k are given and the e_k range over non-zero, pairwise orthogonal projections of rank r_k. For instance, for $n=1$, $\Lambda=\{1\}$ and $R=\{r\}$ we obtain the manifold of projections with a given finite rank r, that was studied in [8]. For the n-uple $\Lambda=(\lambda_1,\cdots,\lambda_n)$ we set $\Lambda^*\colon=(\bar\lambda_1,\cdots,\bar\lambda_n)$. The involution $z\mapsto z^*$ on Z induces a map $M(n,\Lambda,R)\to M(n,\Lambda^*,R)$ where $M(n,\Lambda,R)^*=\{z^*\colon z\in M\}=M(n,\Lambda^*,R)$, and $\Lambda\subset\mathbb{R}$ if and only if $M(n,\Lambda,R)$ consists of hermitian elements.

For a normal algebraic element $a=\sum_{\lambda\in\sigma(a)\setminus\{0\}}\lambda e_\lambda$ we define its *support* to be the projection

$$\mathbf{a}=\text{ supp}\,a\colon=\sum_{\lambda\in\sigma(a)\setminus\{0\}} e_\lambda=e_1+\cdots+e_n.$$

It is clear that $h(\text{ supp}\,(a))=\text{ supp}\,h(a)$ holds for all $h\in\mathsf{Aut}^\circ(Z)$, which combined with the $\mathsf{Aut}^\circ(Z)$-invariance of Peirce projectors P_k gives the following useful formula

$$P_k\big(\text{ supp}\,h(a)\big)=P_k\big(h\,\text{supp}\,(a)\big)=h\,P_k\big(\text{ supp}\,(a)\big)\,h^{-1},\qquad (k=1,1/2,0). \tag{7}$$

Proposition 3.1 *Let $\mathcal{A}$ be the set of all normal algebraic elements of finite rank in Z, and let $M(n,\Lambda,R)$ be defined as in (6). Then*

$$\mathcal{A}=\bigcup_{n,\,\Lambda,\,R} M(n,\,\Lambda,\,R) \tag{8}$$

is a disjoint union of $\mathsf{Aut}^\circ(Z)$-invariant connected subset of Z on which the group $\mathsf{Aut}^\circ(Z)$ acts transitively.

PROOF.
We have seen before that $\mathcal{A}\subset\bigcup_{n,\,\Lambda,\,R}M(n,\,\Lambda,\,R)$. Conversely, let a belong to some $M(n,\,\Lambda,\,R)$ hence we have $a=\sum_k\lambda_k e_k$ for some orthogonal projections e_k. Then $\mathsf{Id}=(e_1+\cdots+e_n)+f$ where f is the projection onto $\ker(a)$ in case $0\in\sigma(a)$ and $f=0$ otherwise. The above properties of the e_k,f yield easily $ap(a)=0$ or $p(a)=0$ according to the cases, where $p\in\mathbb{R}[X]$ is the polynomial $p(z)=(z-\lambda_1).\cdots.(z-\lambda_n)$. Hence $a\in\mathcal{A}$. Clearly (21) is union of disjoint subsets.

Fix one of the sets $M\colon=M(n,\,\Lambda,\,R)$ and take any pair $a,b\in M$. Then

$$a=\lambda_1p_1+\cdots+\lambda_np_n,\qquad b=\lambda_1q_1+\cdots+\lambda_nq_n.$$

In case $0\in\sigma(a)$, set $p_0\colon=\mathsf{Id}-\sum_k p_k$ and $q_0\colon=\mathsf{Id}-\sum_k q_k$. Since rank $p_k=$ rank q_k, the projections p_k and q_k are unitarily equivalent and so are p_0 and q_0. Let us choose orthonormal basis $\mathcal{B}^p_k$ and $\mathcal{B}^q_k$ in the ranges $p_k(H)$ and $q_k(H)$ for $k=0,\,1,\cdots,n$. Then $\bigcup_k\mathcal{B}^p_k$ and $\bigcup_k\mathcal{B}^q_k$ are two orthonormal basis in H. The unitary operator $U\in Z$ that exchanges these basis satisfies $Ua=b$. In particular M is the orbit of any of its points under the action of the unitary group of H. Since this group is connected and its action on Z is continuous, M is connected. □

Let $a\in Z$ be a normal algebraic element with finite rank and $\mathbf{a}=\text{ supp}\,(a)$ its support. In the Peirce decomposition

$$Z=Z_1(\mathbf{a})\oplus Z_{1/2}(\mathbf{a})\oplus Z_0(\mathbf{a})$$

every Peirce space $Z_k(\mathbf{a})_s$ is invariant under the natural involution * of Z, and we let $Z_k(\mathbf{a})_s$ denote its selfadjoint part, $(k=1,1/2,0)$. In what follows, the map $Z\times Z\to Z$ given by $(x,y)\mapsto g(\mathbf{a},x)y$, and the partial maps obtained by fixing one of the variables, will play an important role. For every fixed value $x\in Z_{1/2}(\mathbf{a})$, we get an operator $g(\mathbf{a},x)(\cdot)$ which is an inner JB*-triple derivation of Z, hence we have an operator-valued continuous real-linear map $Z_{1/2}(\mathbf{a})\to\mathsf{Der}\,(Z)$. Moreover $g(\mathbf{a},x)(\cdot)$ is a C*-algebra derivation if and only if $x\in Z_{1/2}(\mathbf{a})_s$ (see 3.3). For $y=a$ fixed, we get the map $x\mapsto g(\mathbf{a},x)a$ for which we introduce the notation

$$\Phi_a(x)\colon=g(\mathbf{a},x)a=\{\mathbf{a}\,x\,a\}-\{x\,\mathbf{a}\,a\}=\big(Q(\mathbf{a},a)-L(\mathbf{a},a)\big)x,\qquad x\in Z.$$

First we discuss $Z_{1/2}(\mathbf{a})$.

Proposition 3.2 *Let $a \in Z$ be a normal algebraic element of finite rank, and let $\mathbf{a} = e_1 + \cdots + e_n$ be its support. Then $Z_{1/2}(\mathbf{a})$ consists of the operators*

$$u = \sum_k u_k, \quad u_k \in Z_{1/2}(e_k), \quad e_k\, u_j = u_j\, e_k = 0, \quad j \neq k, \quad (1 \leq j,\, k \leq n). \tag{9}$$

If $u \in Z_{1/2}(\mathbf{a})_s$, then we have the additional condition $u_k \in Z_{1/2}(e_k)_s$.

PROOF.
Let $u \in Z$ be selfadjoint. The relation $u \in Z_{1/2}(\mathbf{a})$ is equivalent to $u = 2\{\mathbf{a}\mathbf{a}u\}$ which now reads

$$u = 2\{\mathbf{a}\mathbf{a}u\} = \mathbf{a}\mathbf{a}^* u + u\mathbf{a}^*\mathbf{a} = \sum_k (e_k u + u e_k) = \sum_k u_k$$

where

$$u_k \colon = e_k u + u e_k \quad \text{for} \quad 1 \leq k \leq n. \tag{10}$$

Note that $e_j, e_k \in Z_1(\mathbf{a})$, hence by the Peirce multiplication rules $\{e_j u e_k\} \in \{Z_1(\mathbf{a}) Z_{1/2}(\mathbf{a}) Z_1(\mathbf{a})\} = \{0\}$, that is $e_j u e_k + e_k u e_j = 0$ for all $1 \leq j,\, k \leq n$. Multiplying the latter by e_j with $j \neq k$ yields $e_j u e_k = 0$ for $j \neq k,\ (1 \leq j,\, k \leq n)$. Therefore by (10),

$$\begin{aligned} 2\{e_k e_k u_k\} = e_k(e_k u + u e_k) \quad &+ (e_k u + u e_k) e_k = \\ (e_k u + u e_k) + 2 e_k u e_k \quad &= (e_k u + u e_k) = u_k \end{aligned}$$

which shows $u_k \in Z_{1/2}(e_k)$ and clearly $u_k = u_k^*$ for $1 \leq k \leq n$. Multiplying in (10) by e_j with $j \neq k$ we get $u_k e_j = e_j u_k = 0$ and in particular $e_j \Box u_k = u_k \Box e_j = 0$ for $j \neq k$.

Conversely, let u_k satisfy the properties in (9). Then $u \colon = \sum_k u_k$ is selfadjoint and $e_k u = e_k \left(\sum_j u_j\right) = e_k u_k$. Similarly$u e_k = u_k e_k$, hence $2\{\mathbf{a}\mathbf{a}u\} = \mathbf{a}\mathbf{a}^* u + u\mathbf{a}^*\mathbf{a} = \left(\sum_j e_j\right) u + u \left(\sum_j e_j\right) = \sum_j (e_j u + u e_j) = u$,.

Using the *-invariance of $Z_{1/2}(\mathbf{a})$ every element in this space can be written in the form $u = u_1 + i u_2$ with $u_1, u_2 \in Z_{1/2}(\mathbf{a})_\mathbf{s}$ and the result follows easily. □

The following result should be compared with ([1], th. 3.1)

Proposition 3.3 *Let $a \in Z$ be a normal algebraic element of finite rank and $\mathbf{a} \colon = \operatorname{supp}(a)$. Then for any $u \in Z_{1/2}(\mathbf{a})$, the operator $g(\mathbf{a}, u) \colon = \mathbf{a} \Box u - u \Box \mathbf{a}$ is an inner C*-derivation of Z if and only if u is selfadjoint.*

PROOF.
Let $a = \sum_k \lambda_k e_k$ and $\mathbf{a} = \sum_k e_k$ be the spectral resolution and the support of a. Suppose $u = u^*$. By (3.2) u has the form $u = \sum u_k$ with $u_k \in Z_{1/2}(e_k)_s$ and $e_k \Box u_j = u_j \Box e_k = 0$ for all $j \neq k$. Therefore

$$g(\mathbf{a}, u) = \sum_k (e_k \Box u_k - u_k \Box e_k) = \sum_k g(e_k, u_k). \tag{11}$$

Here the e_k are projections in Z and $u_k \in Z_{1/2}(e_k)_s$, hence by ([1], th. 3.1) each $g(e_k, u_k)$ is an inner C*-derivation of Z and so is the sum. Conversely, since $\mathbf{a}$ is a projection, whenever $g(\mathbf{a}, u)$ is a C*-algebra derivation we have $u \in Z_{1/2}(\mathbf{a})_\mathbf{s}$ again by ([1], th. 3.1). □

Now consider the joint Peirce decomposition of Z relative to the family $(e_1, \cdots, e_n)$ where $a = \lambda_1 e_1 + \cdots + \lambda_n e_n$ is the spectral resolution of a. Remark that $\bigoplus_{1 \leq k \leq n} i\, A(e_k) \subset Z_1(\mathbf{a})$ is a direct summand of Z, hence so is the space

$$X \colon = \Big(\bigoplus_{1 \leq k \leq n} i\, A(e_k) \Big) \oplus Z_{1/2}(\mathbf{a}).$$

Proposition 3.4 *Let $a \in Z$ be a normal algebraic element of finite rank and $\mathbf{a} \colon = \operatorname{supp}(a)$. Then Φ_a is a surjective complex linear homeomorphism of $Z_{1/2}(\mathbf{a})$. If a is hermitian then Φ_a is a surjective real linear homeomorphism of X that preserves the subspace $\bigoplus_{1 \leq k \leq n} i\, A(e_k)$.*

PROOF.
Let $x = iv + u \in X$ where $v \in \bigoplus_{1\leq k\leq n} A(e_k)$ and $u \in Z_{1/2}(\mathbf{a})$. The Peirce multiplication rules give for $v = \sum_j v_j$ with $v_j \in A(e_j)$ and $u = \sum_k u_k$ according to (3.2)

$$\begin{aligned}
&\{\mathbf{a} Z_{1/2}(\mathbf{a}) a\} = \{0\},\\
&\{\mathbf{a}\, iv\, a\} = -i\{\sum_j e_j \sum_k v_k \sum_l \lambda_l e_l\} = -i\sum_k \lambda_k v_k,\\
&\{u\, \mathbf{a}\, a\} = i\{\sum_j u_j \sum_k e_k \sum_l \lambda_l e_l\} = \frac{i}{2}\sum_k \lambda_k u_k.
\end{aligned}$$

Therefore

$$\Phi_a(x) = -2i\sum_k \lambda_k v_k - \frac{1}{2}\sum_k \lambda_k u_k \in \big(\bigoplus_{1\leq k\leq n} Z_1(e_k)\big) \oplus Z_{1/2}(\mathbf{a}). \tag{12}$$

It is now clear that Φ_a preserves $Z_{1/2}(\mathbf{a})$. If a is hermitian then $\Lambda \subset \mathbb{R}^n$ and Φ_a also preserves $\bigoplus_{1\leq k\leq n} i\, A(e_k)$. Moreover $\Phi_a(x) = 0$ with $x \in X$ is equivalent to $\sum \lambda_k v_k = 0 = \sum \lambda_k u_k$ which is equivalent to $v = 0 = u$ since the coefficients satisfy $\lambda_k \in \sigma(a)\backslash\{0\}$. We can recover x from $\Phi_a(x)$, hence the result follows. □

Recall that a subset $M \subset Z$ is called a *real analytic* (respectively, *holomorphic*) submanifold if to every $a \in M$ there are open subsets $P, Q \subset Z$ and a closed real-linear (resp. complex) subspace $X \subset Z$ with $a \in P$ and $\phi(P \cap M) = Q \cap X$ for some bianalytic (resp. biholomorphic) map $\phi\colon P \to Q$. If to every $a \in M$ the linear subspace $X = T_a M$, called the *tangent space* to M at a, can be chosen to be topologically complemented in Z then M is called a *direct submanifold* of Z.

Fix one of the sets $M = M(n, \Lambda, R)$ and a point $a \in M$ with spectral resolution $a = \sum_k \lambda_k e_k$. By the orthogonality properties of the e_k, the successive powers of a have the expression

$$a^l = \lambda_1^l e_1 + \cdots + \lambda_n^l e_n, \qquad 1 \leq l \leq n,$$

where the determinant $\det(\lambda_k^l) \neq 0$ does not vanish since it is a Vandermonde determinant and the λ_k are pairwise distinct. Thus the e_k are polynomials in a whose coefficients are rational functions of the λ_k. Suppose M is a differentiable manifold, and let us obtain its tangent space $T_a M$. Consider a smooth curve $t \mapsto a(t)$ through $a \in M$, $t \in I$, for a neighbourhood I of $0 \in \mathbb{R}$ and $a(0) = a$. Each $a(t)$ has a spectral resolution

$$a(t) = \lambda_1 e_1(t) + \cdots + \lambda_n e_n(t),$$

therefore the maps $t \mapsto e_k(t)$, $(1 \leq k \leq n)$, are smooth curves in the manifolds $\mathfrak{M}(r_k)$ of the projections in Z that have fixed finite rank $r_k = \text{rank }(e_k)$, whose tangent spaces at $e_k = e_k(0)$ are $Z_{1/2}(e_k)$ (see [1] or [8]). Therefore

$$u_k\colon = \frac{d}{dt}|_{t=0} e_k(t) \in Z_{1/2}(e_k), \qquad 1 \leq k \leq n.$$

Since the spectral projections of $a(t)$ corresponding to different spectral values $\lambda_k \neq \lambda_j$ are orthogonal, we have $e_j(t)\, e_k(t) = 0$ for all $t \in I$, and taking the derivative at $t = 0$,

$$e_j\, u_k = u_k\, ej = 0, \qquad j \neq k,\ \ 1 \leq j, k \leq n. \tag{13}$$

By 19, the tangent vector to $t \mapsto a(t)$ at $t = 0$, that is, $u\colon = \frac{d}{dt}|_{t=0} a(t) = \sum_k \lambda_k u_k$ satisfies

$$\{\mathbf{a}\,\mathbf{a}\, u\} = \{\sum_j e_j \sum_k e_k \sum_l \lambda_l u_l\} = \sum_{j,k,l} \lambda_l \{e_j e_k u_l\} =$$
$$\sum_{k,l} \lambda_l \{e_k e_k u_l\} = \frac{1}{2}\sum_{k,l} \lambda_l (e_k u_l + u_l e_k) = \sum_l \lambda_l \{e_l e_l u_l\} = \frac{1}{2}\sum_l \lambda_l u_l = \frac{1}{2} u$$

hence $u \in Z_{1/2}(\mathbf{a})$, and $T_a M$ can be identified with a vector subspace of $Z_{1/2}(\mathbf{a})$. In fact $T_a M = Z_{1/2}(\mathbf{a})$ as it easily follows from the following result that should be compared with ([1] th. 3.3)

Theorem 3.5 *The sets $M = M(n, \Lambda, R)$ defined in (6) are holomorphic direct submanifolds of Z. The tangent space at the point $a \in M$ is the Peirce subspace $Z_{1/2}(\mathbf{a})$ where $\mathbf{a} =$ supp (a), and a local chart at a given by*

$$f\colon u \mapsto f(u)\colon = (\exp g(\mathbf{a}, u))a \qquad (14)$$

with $g(\mathbf{a}, u) = \mathbf{a}\Box u - u\Box\mathbf{a}$.

PROOF.
$M \subset Z$ is invariant under $\mathsf{Aut}^\circ(Z)$. Fix any $a \in M$ and let $X\colon = \big(\bigoplus_{1\le k\le n} i\, A(e_k)\big) \oplus Z_{1/2}(\mathbf{a})$. Thus $Z = X \oplus Y$ for a certain subspace Y. The mapping $X \oplus Y \to Z$ defined by $(x, y) \mapsto F(x, y)\colon = (\exp g(\mathbf{a}, x))y \in Z$ is a real-analytic and its Fréchet derivative at $(0, a)$ is invertible. In fact this derivative is

$$\begin{aligned} \frac{\partial F}{\partial x}|_{(0,a)}(u, v) &= g(\mathbf{a}, u)a = \Phi_a(u), \\ \frac{\partial F}{\partial y}|_{(0,a)}(u, v) &= \big(\exp g(\mathbf{a}, 0)\big)v = v, \end{aligned}$$

which is invertible according to (3.4). By the implicit function theorem there are open sets U, V with $0 \in U \subset X$ and $a \in V \subset Y$ such that $W\colon = F(U \times V)$ is open in Z and $F\colon U \times V \to W$ is bianalytic.

To simplify notation set $X_1 = Z_{1/2}(\mathbf{a}) \subset X$. Then $f = F|X_1$ establishes a real analytic homeomorphism between the sets $N_1\colon = U \cap X_1$ and $M_1\colon = f(N_1)$. Since X_1 is a direct summand in X (hence also in Z), the image $M_1 = f(N_1)$ is a direct submanifold.

The operator $g(\mathbf{a}, x) = \mathbf{a}\Box x - x\Box\mathbf{a}$ is an inner JB*triple derivation of Z, hence $h\colon = \exp g(\mathbf{a}, u)$ is a JB*-triple automorphism of Z. Actually h lies in $\mathsf{Aut}^\circ(Z)$, the identity connected component. But it is known ([10]) that $\mathsf{Aut}\,(Z)$ has two connected components and that the elements in the identity component are C*-algebra automorphisms of Z since they have the form $z \mapsto UzU^*$ for some U in the unitary group of H. In particular h preserves normality, spectral values and ranks hence it preserves M and so

$$M_1 = f(N_1) = \{(\exp g(\mathbf{a}, u))a : u \in N_1\} \subset M.$$

To complete the proof, it suffices to show that $f = F|X_1$ is a biholomorphic mapping. The Fréchet derivative of f at a is

$$f'|_a(u) = g(\mathbf{a}, u)a = \{\mathbf{a},\, u,\, a\} - \{u,\, \mathbf{a},\, a\}, \qquad u \in Z_{1/2}(\mathbf{a}).$$

Therefore $\overline{\partial} f'u = \{\mathbf{a},\, u,\, a\}$ and $\partial f'u = -\{u,\, \mathbf{a},\, a\}$ are the (uniquely determined) complex-linear and complex-antilinear components of $f'u$. The Peirce rules give $\{\mathbf{a},\, u\, a\} = 0$ for all $u \in Z_{1/2}(\mathbf{a})$, hence f is holomorphic and the same argument holds for the inverse f^{1} map. □

Remark that if the algebraic element a is a projection then $\mathbf{a} = a$ and M as a differentiable manifold is the one constructed in ([1] th. 3.3) and [8].

4 The Jordan connection on $M(n,\, \Lambda,\, R)$

Let $a \in M\colon = M(n,\, \Lambda,\, R)$ and set $\mathbf{a} =$ supp (a). Recall that a vector field X on M is a map from M to the tangent bundle TM. Thus X_a, the value of X at $a \in M$, satisfies $X_a \in T_aM \approx Z_{1/2}(\mathbf{a})$. We let $\mathfrak{D}(M)$ be the Lie algebra of smooth vector fields on M. Since the tangent space T_aM at $a \in M$ has been identified with $Z_{1/2}(\mathbf{a})$, we shall consider every vector field on M as a Z-valued function such that the value at a is contained in $Z_{1/2}(\mathbf{a})$. Let Y'_a be the Fréchet derivative of $Y \in \mathfrak{D}(M)$ at a. Thus Y'_a is a bounded linear operator $Z_{1/2}(\mathbf{a}) \to Z$, hence $Y'_aX_a \in Z$ and it makes sense to take the projection $P_{1/2}(\mathbf{a})Y'_aX_a \in Z_{1/2}(\mathbf{a}) \approx T_aM$.

Definition 4.1 *We define a connection ∇ on M by*

$$(\nabla_X Y)_a\colon = P_{1/2}(\,\text{supp}\,(a))\, Y'_a X_a, \qquad X,\, Y \in \mathfrak{D}(M), \qquad a \in M.$$

Note that if a is a projection, then $a =$ supp (a) and ∇ coincides with the affine connection defined in ([1] def 3.6) and [8]. It is a matter of routine to check that ∇ is an affine connection on M, that it is $\mathsf{Aut}^\circ(Z)$- invariant and torsion-free, i. e.,

$$g\,(\nabla_X Y) = \nabla_{g(X)}\, g\,(Y), \qquad g \in \mathsf{Aut}^\circ(Z),$$

where $(g\,X)_a\colon = g'_a\,(X_{g_a^{-1}})$ for all $X \in \mathfrak{D}(M)$, and

$$T(X, Y)\colon = \nabla_X Y - \nabla_Y X - [XY] = 0, \qquad X, Y \in \mathfrak{D}(M).$$

Theorem 4.2 *Let the manifolds M be defined as in (6). Then the ∇-geodesics of M are the curves*

$$\gamma(t) \colon = (\exp\, t\, g(\mathbf{a}, u))a, \qquad t \in \mathbb{R}, \tag{15}$$

where $a \in M$ and $u \in Z_{1/2}(\mathbf{a})$.

PROOF.
Recall that the geodesics of ∇ are the curves $t \mapsto \gamma(t) \in M$ that satisfy the second order ordinary differential equation

$$\big(\nabla_{\dot\gamma(t)}\, \dot\gamma(t)\big)_{\gamma(t)} = 0.$$

Let $u \in Z_{1/2}(\mathbf{a})$. Then $g(\mathbf{a}, u) = \mathbf{a}\Box u - u\Box\mathbf{a}$ is an inner JB*-triple derivation of Z, and, as established in the proof of (3.5), $h(t) \colon = \exp\, t\, g(\mathbf{a}, u)$ is an inner C*-automorphism of Z. Thus $h(t)a \in M$ and $t \mapsto \gamma(t)$ is a curve in the manifold M. Clearly $\gamma(0) = a$ and taking the derivative with respect to t at $t = 0$ we get by the Peirce rules

$$\begin{aligned}
\dot\gamma(t) &= g(\mathbf{a}, u)\gamma(t) = h(t)g(\mathbf{a}, u)a, & \dot\gamma(0) &= g(\mathbf{a}, u)a \in Z_{1/2}(\mathbf{a}),\\
\ddot\gamma(t) &= g^2(\mathbf{a}, u)\gamma(t) = h(t)g(\mathbf{a}, u)^2 a, & \ddot\gamma(0) &= g(\mathbf{a}, u)\dot\gamma(0) \in Z_1(\mathbf{a}) \oplus Z_0(\mathbf{a}).
\end{aligned}$$

In particular $P_{1/2}(\mathbf{a})g(\mathbf{a}, u)^2 a = 0$. The definition of ∇ and the relation (7) give

$$\begin{aligned}
\big(\nabla_{\dot\gamma(t)}\, \dot\gamma(t)\big)_{\gamma(t)} &= P_{1/2}(\operatorname{supp}\gamma(t))\Big(\dot\gamma(t)'_{\gamma(t)}\, \dot\gamma(t)\Big) = P_{1/2}(\operatorname{supp}\gamma(t))\,\ddot\gamma(t) = \\
&P_{1/2}\big(\operatorname{supp} h(t)a\big)\, h(t)g(\mathbf{a}, u)a = h(t)P_{1/2}\big(\operatorname{supp}(a)\big)\, g(\mathbf{a}, u)^2 a = 0
\end{aligned}$$

for all $t \in \mathbb{R}$. Using the representation $u = \sum_k u_k$ given by (9) one gets $g(\mathbf{a}, u)a = -\frac{1}{2}\sum_k \lambda_k u_k$, and as $\lambda \in \sigma(a)\backslash\{0\}$ the mapping $u \mapsto g(\mathbf{a}, u)a$ is a linear homeomorphism of $Z_{1/2}(\mathbf{a})$. Since geodesics are uniquely determined by the initial point $\gamma(0) = a$ and the initial velocity $\dot\gamma(0) = g(\mathbf{a}, u)a$, the above shows that family of curves in (15) with $a \in M$ and $u \in T_aM \approx Z_{1/2}(\mathbf{a})$ are all geodesics of the connection ∇. □

Recall that $\mathbf{a} = \operatorname{supp}(a)$ is a finite rank projection, hence by ([8], th. 1.1) the JB*-subtriple $Z_{1/2}(\mathbf{a})$ has finite rank and the tangent space $T_aM \approx Z_{1/2}(\mathbf{a})$ is linearly homeomorphic to a Hilbert space under an $\mathsf{Aut}^\circ(Z)$-invariant scalar product (say $\langle\cdot\,,\cdot\rangle$). Thus we can define a Riemann metric on M by

$$g_a(X, Y) \colon = \langle X_{\mathbf{a}}, Y_{\mathbf{a}}\rangle, \qquad X, Y \in \mathfrak{D}(M), \qquad a \in M. \tag{16}$$

Remark that g is *hermitian*, i.e. we have $g_a(iX,\, iY) = g_a(X, Y)$, and that it has been defined in algebraic terms, hence it is $\mathsf{Aut}^\circ(Z)$-invariant. Moreover, ∇ is compatible with the Riemann structure, i. e.

$$X\, g(Y, W) = g(\nabla_X\, Y,\, W) + g(Y,\, \nabla_X\, W), \qquad X, Y, W \in \mathfrak{D}(M).$$

Therefore, ∇ is the only Levi-Civita connection on M. On the other hand, let the map $J \colon Z_{1/2}(\mathbf{a}) \to Z_{1/2}(\mathbf{a})$ be given by $Jz \colon = iz$. Clearly $J^2 = -\mathsf{Id}$, hence J defines (the usual) complex structure on the tangent space to M and ∇ is J-hermitian

$$\nabla_X\,(iY) = i\nabla_X\, Y, \qquad X, Y \in \mathfrak{D}(M),$$

hence ∇ is the only hermitian connection on M. Thus the Levi-Civita and the hermitian connection are the same in this case, and so ∇ is the Kähler connection on M.

For a tripotent $e \in \mathsf{Tri}\,(Z)$, the Peirce reflection around e is the linear map $S_e \colon = \mathsf{Id} - P_{1/2}(e)$ or in detail $z = z_1 + z_{1/2} + z_0 \mapsto S_e(z) = z_1 - z_{1/2} + z_0$ where z_k are the Peirce e-projections of z, $(k = 1, 1/2, 0)$. Recall that S_e is an involutory triple automorphism of Z with $S_e(e) = e$, and that if e is a projection (taken as a tripotent) then S_e is a C*-algebra automorphism of Z. This applies to $\mathbf{a} = \operatorname{supp}(a)$, hence to each $a \in M$ we get $S_{\mathbf{a}}$, an involutory automorphism of the manifold M which in this way becomes a symmetric holomorphic Riemann (Kähler) manifold. Note that in general $\mathbf{a} \notin M$ even if $a \in M$, hence $S_{\mathbf{a}}$ may have no fixed points in M.

It would be interesting to know if any two points a, b in M can be joined by a geodesic and whether geodesics are minimizing curves for the Riemann distance. The answers to these questions are affirmative when M consists of projections of the same finite rank (see [8]).

5 Algebraic elements in JB*-triples

The role that projections play in the study of algebras is taken by tripotents in the study of triple systems. A spectral calculus and a notion of algebraic elements is available in the stetting of JB*-triples. In what follows we shall consider the manifold of all finite rank algebraic elements in a JB*-triple Z.

Definition 5.1 *An element $a \in Z$ is called algebraic if there exits a decomposition*

$$a = \lambda_1 e_1 + \cdots + \lambda_n e_n \tag{17}$$

where (e_k) is a family of pairwise orthogonal tripotents in Z and (λ_k) are complex coefficients.

For an algebraic element $a \in Z$ the above decomposition can always be chosen in such a way that every e_k is non-zero and the λ_k are real numbers with $0 < \lambda_1 < \cdots \lambda_n$, and under these additional conditions the spectral representation of a is unique. Clearly a has finite rank if and only if every so does every e_k.

Remark that for $Z = \mathcal{L}(H)$, normal algebraic elements in the C*-algebra Z are algebraic elements in Z as a JB*-triple. Given a positive integer $n \in \mathbb{N}$, an increasing n-uple of non-zero real numbers $\Lambda = (\lambda_1, \cdots, \lambda_n)$ and an n-uple $R = (r_1, \cdots, r_n)$ where $0 < r_k \in \mathbb{N}$, we define

$$N(n,\, \Lambda,\, R) \colon = \{ \sum_k \lambda_k e_k \, : \; e_j \Box e_k = 0 \;\text{ for } j \neq k, \;\; \text{rank } (e_k) = r_k, \; 1 \leq j,\, k \leq n \} \tag{18}$$

to be the set of the elements (17) where the coefficients λ_k and ranks r_k are given and the e_k range over non-zero, pairwise orthogonal tripotents in Z such that rank $(e_k) = r_k$. The set $\mathcal{A}$ of finite rank algebraic elements in Z is the disjoint union $\mathcal{A} = \cup_{n,\,\Lambda,\,R} N(n,\, \Lambda,\, R)$.

Lemma 5.2 *Let Z be an irreducible JBW*-triple. Then each of sets $N = N(, n\,\Lambda,\, R)$ is an Aut$^\circ(Z)$-invariant connected subset of Z on which the group Aut$^\circ(Z)$ acts transitively.*

PROOF.
Irreducible JBW*-triples are Cartan factors and we may assume that Z is a not *special* as otherwise $\dim Z < \infty$ and the result is known [16]. Thus Z is a J*-algebra in the sense of Harris [4] that is, a weak*-operator closed complex linear subspace of $\mathcal{L}(H, K)$ that is closed under the operation of taking triple products, for suitable complex Hilbert spaces H, K with $\dim H \leq \dim K$. Tripotents are the partial isometries $e \colon H \to K$ that lie in Z.

We make a type by type proof. Let $Z = \mathcal{L}(H, K)$ be a type I Cartan factor and let $a, b \in N$. In particular

$$a = \lambda_1 e_1 + \cdots + \lambda_n e_n, \qquad b = \lambda_1 e'_1 + \cdots + \lambda_n e'_n$$

Let $H_k, H'_k \subset H$ be the domains of the partial isometries e_k and e'_k, and similarly let $K_k, K'_k \subset K$ denote their respective ranges. Since e_k and e'_k have the same finite rank r_k, they are unitarily equivalent, that is there are unitary operators $U_k \colon H_k \to H'_k$ and $V_k \colon K_k \to K_k$ such that $e'_k = V_k e_k U_k$. Since the e_k are pairwise orthogonal we have $H_k \perp H_j$ and $K_k \perp K_j$ for $k \neq j$ and $\bigoplus U_k$, $\bigoplus V_k$ are unitary operators on $\bigoplus H_k$ and $\bigoplus K_k$ that can be extended to unitary operators $U \colon H \to H$ and $V \colon K \to K$ if needed. The mapping $Z \to Z$ given by $z \mapsto VzU$ is a JB*-triple automorphism that lies in Aut$^\circ(Z)$ [10] and clearly satisfies $b = VaU$. Hence Aut$^\circ(Z)$ acts transitively on N, N is connected and invariant under that group.

Cartan factors of types II and III can treated in the same way. The case of spin factors may be discussed with a different approach, but we shall not go into details. □

Now consider the joint Peirce decomposition of Z relative to the family $(e_1, \cdots, e_n)$ where $a = \lambda_1 e_1 + \cdots + \lambda_n e_n$ is the spectral resolution of a. Let the *support* of a be tripotent $\mathbf{a} =$ supp $a \colon = e_1 + \cdots + e_n$, and note that

$$X \colon = \big(\bigoplus_{1 \leq k \leq n} i\, A(e_k) \big) \oplus Z_{1/2}(\mathbf{a}).$$

is a topologically complemented subspace in Z.

Fix one of the sets $N = N(n, \Lambda, R)$ and a point $a \in N$ with spectral resolution $a = \sum_k \lambda_k e_k$. From the properties $e_k \Box e_j = 0$ for $j \neq k$, the successive odd powers of a have the expression

$$a^l = \lambda_1^{2l+1} e_1 + \cdots + \lambda_n^{2l+1} e_n, \qquad 0 \leq l \leq n-1,$$

where the determinant $\det(\lambda_k^{2l+1}) \neq 0$ does not vanish since it is a Vandermonde determinant and the λ_k are pairwise distinct. Thus the e_k are polynomials in a whose coefficients are rational functions of the λ_k. Suppose N is a differentiable manifold, and let us obtain its tangent space T_aN. Consider a smooth curve $t \mapsto a(t)$ in N through a, $t \in I$, for a neighbourhood I of $0 \in \mathbb{R}$ and $a(0) = a$. Each $a(t)$ has a spectral resolution

$$a(t) = \lambda_1 e_1(t) + \cdots + \lambda_n e_n(t),$$

therefore the maps $t \mapsto e_k(t)$, $(1 \leq k \leq n)$, are smooth curves in the manifolds $\mathfrak{N}(r_k)$ of the tripotents in Z that have fixed finite rank $r_k = \text{rank }(e_k)$, whose tangent spaces at $e_k = e_k(0)$ are respectively $i\,A(e_k) \oplus Z_{1/2}(e_k)$ (see [1] or [8]). Therefore

$$z_k \colon = \frac{d}{dt}|_{t=0}\, e_k(t) = iv_k + u_k \colon \in i\,A(e_k) \oplus Z_{1/2}(e_k), \qquad 1 \leq k \leq n.$$

Set $v \colon = \sum_k \lambda_k v_k$ and $u \colon = \sum_k \lambda_k u_k$. From $Z_1(e_k) \Box Z_0(e_j) = \{0\}$, we get

$$\{\mathbf{a}\,\mathbf{a}\,iv\} = i \sum_{j,k,l} \lambda_l \{e_j e_k v_l\} = i \sum_k \lambda_k v_k = iv \in i \bigoplus_k A(e_k)$$

The spectral tripotents of $a(t)$ corresponding to different spectral values $\lambda_k \neq \lambda_j$ are orthogonal, hence $e_j(t) \Box e_k(t) = 0$ for all $t \in I$, and taking the derivative at $t = 0$ we get

$$e_j \Box u_k = u_k \Box ej = 0, \qquad j \neq k, \;\; 1 \leq j, k \leq n. \tag{19}$$

Hence

$$\{\mathbf{a}\,\mathbf{a}\,u\} = \{\sum_j e_j \sum_k e_k \sum_l \lambda_l u_l\} = \sum_{j,k,l} \lambda_l \{e_j e_k u_l\} = \frac{1}{2} \sum_k \lambda_k u_k = \frac{1}{2} u$$

which shows that $u \in Z_{1/2}(\mathbf{a})$. By 19, the tangent vector to $t \mapsto a(t)$ at $t = 0$ is $z \colon = \frac{d}{dt}|_{t=0} a(t) = \sum_k \lambda_k (iv_k + u_k) = iv + u$ hence it satisfies

$$\{\mathbf{a}\,\mathbf{a}\,z\} = iv + \frac{1}{2} u \in i \bigoplus A(e_k) \oplus Z_{1/2}(\mathbf{a}),$$

hence T_aN can be identified with a vector subspace of $i \bigoplus A(e_k) \oplus Z_{1/2}(\mathbf{a})$. In fact T_aN coincides with that space as it easily follows from the following result that should be compared with ([1] th. 3.3)

Theorem 5.3 *The sets $N = N(n, \Lambda, R)$ defined in (18) are real analytic direct submanifolds of Z. The tangent space at the point $a \in N$ is the Peirce subspace X, where $\mathbf{a} = \text{supp}\,(a)$, and a local chart at a given by*

$$f \colon z \mapsto f(z) \colon = (\exp g(\mathbf{a}, z))a \tag{20}$$

with $g(\mathbf{a}, z) = \mathbf{a} \Box z - z \Box \mathbf{a}$.

PROOF.
$N \subset Z$ is invariant under $\mathsf{Aut}^\circ(Z)$. Fix any $a \in N$ and let $X \colon = \big(\bigoplus_{1 \leq k \leq n} i\,A(e_k)\big) \oplus Z_{1/2}(\mathbf{a})$. Thus $Z = X \oplus Y$ for a certain subspace Y. The mapping $X \oplus Y \to Z$ defined by $(x, y) \mapsto F(x, y) \colon = (\exp g(\mathbf{a}, x))y \in Z$ is a real-analytic and its Fréchet derivative at $(0, a)$ is invertible as proved in (3.4). By the implicit function theorem there are open sets U, V with $0 \in U \subset X$ and $a \in V \subset Y$ such that $W \colon = F(U \times V)$ is open in Z and $F \colon U \times V \to W$ is bianalytic and the image $F(U)$ is a direct real analytic submanifold of Z.

The operator $g(\mathbf{a}, z) = \mathbf{a} \Box z - z \Box \mathbf{a}$ is an inner JB*triple derivation of Z, hence $h \colon = \exp g(\mathbf{a}, z)$ is a JB*-triple automorphism of Z. Actually h lies in $\mathsf{Aut}^\circ(Z)$, the identity connected component. In particular h preserves the algebraic character and the spectral decomposition, hence it preserves N and so

$$F(N) = \{(\exp g(\mathbf{a}, z))a : z \in U\} \subset N.$$

This completes the proof. □

Definition 5.4 *For the tripotents e, e' we set $e \sim e'$ if and only if e and e' have the same k-Peirce projectors for $k = 0, 1/2, 1$.*

This notion was introduced by Neher who proved ([17], th.2.3) that

$$e \sim e' \iff e \in Z_1(e') \text{ and } e' \in Z_1(e), \tag{21}$$

or equivalently if and only if $e \square e = e' \square e'$. Next we extend this relation to an equivalence in the manifold N.

Definition 5.5 *Let a, b be elements in N with spectral resolutions $a = \sum_k \lambda_k e_k$ and $b = \sum_k \lambda_k f_k$ respectively. We say that a and b are equivalent (and write $a \sim b$) if the joint Peirce decompositions of Z relative to the orthogonal families $\mathcal{E} = (e_k)$ and $\mathcal{F} = (f_k)$ are the same.*

Note that $\sim$ coincides with the equivalence of Neher when the algebraic elements a and b are tripotents. By ([16], th. 3.14), the Peirce spaces of the tripotent e_k can be expressed in terms of the joint Peirce decomposition of Z relative to $\mathcal{E}$, hence $a \sim b$ if and only if $e_k \sim f_k$ for $1 \leq k \leq n$.

Proposition 5.6 *Let a, b be points in N such that $a = \sum \lambda_k e_k$ and $b = (\exp g(\mathbf{a}, z)a$ for some tangent vector $z = iv + u \in \big(\bigoplus_{1 \leq k \leq n} i\, A(e_k)\big) \oplus Z_{1/2}(\mathbf{a})$. Then $a \sim b$ if and only if $u = 0$.*

PROOF.
Let $b = (\exp g(\mathbf{a}, z)a = \sum_k \lambda_k f_k$ be the spectral resolution of b. Then each f_k is an odd polynomial in b, say $f_k = p_k(b)$, $1 \leq k \leq n$. To simplify the notation, consider the index $k = 1$ and omit the reference to it in the rest of the proof. If $a \sim b$ then $e \sim f$ hence by (21) we must have $f = \{eef\}$ that is

$$p(b) = \{eep(b)\} = p(\{eeb\}) \tag{22}$$

Clearly we have $\rho b \sim a$ for all $\rho \in \mathbb{T}$, which replaced above yields an identity between two polynomials in ρ. Let X^m, for some positive odd integer m, be the term of p of lowest degree whose coefficient is not zero. Then (22) entails $b^m = \{eeb^m\}$, that is $(\exp g(\mathbf{a}, z))^m a = \{e\, e\, (\exp g(\mathbf{a}, z))^m a\}$. Taking the Fréchet derivative at the origin $g(\mathbf{a}, \cdot)\, a = \{e\, e\, g(\mathbf{a}, \cdot)\, a\}$, which evaluated at the tangent vector $z = iv + u = i \sum_k v_k + \sum_k u_k$ and using the Peirce rules as in the proof of (3.4) yields $u = 0$. The converse is easy. □

In particular, there is a neighbourhood of a in N in which the algebraic elements b equivalent to a are those of the form $b = (\exp g(\mathbf{a},\, iv))\, a$ with $v = \sum_k v_k \in \bigoplus_k A(e_k)$, which gives the expression of the fibre of N through a.

Proposition 5.7 *Let $a \in N$ be an algebraic element in Z with spectral resolution $a = \sum_k \lambda_k e_k$. Then the fibre of N through a is the set of the elements $\sum_k \lambda_k z_k$ where z_k lies in the unit circle of the JB*-algebra $Z_1(e_k)$ for $1 \leq k \leq n$.*

PROOF.
Let $v = \sum_k \lambda_k v_k \in \bigoplus_k A(e_k)$, and consider the curves in Z

$$\phi(t) \colon = (\exp\ tg(\mathbf{a}, iv))a, \qquad \psi(t) \colon = \sum_k \lambda_k (\exp\ tg(e_k, iv_k))e_k \colon = \sum_k \lambda_k \psi_k(t), \qquad t \in \mathbb{R}.$$

They are the solutions of the differential equations

$$\frac{d\phi(t)}{dt} = g(\mathbf{a}, \phi(t)), \qquad \frac{d\psi(t)}{dt} = \sum_k \lambda_k g(e_k, \psi_k(t))$$

with the initial conditions $\phi(0) = a$ and $\psi(0) = \sum_k \lambda_k e_k = a$ respectively. From $Z_1(e_k) \square Z_1(e_j) = \{0\}$ for $k \neq j$ we get

$$g(\mathbf{a}, iv) = g(\sum_k e_k,\, i \sum_j \lambda_j v_j) = \sum_k \lambda_k g(e_k,\, iv_k)$$

and the uniqueness of solutions of differential equations gives $\phi(t) = \sum_k \lambda_k \psi_k(t)$ for all $t \in \mathbb{R}$. But it is known ([16] th. 5.6) that for fixed k, $1 \leq k \leq n$, the set $z_k = (\exp tg(e_k, iv_k))e_k$, $t \in \mathbb{R}$, $v_k \in A(e_k)$, is the unit circle of the JB*-algebra $Z_1(e_k)$, that is the set of those $w \in Z_1(e_k)$ that satisfy $w^* = w^{-1}$. This completes the proof. □

By restricting the local charts in (20) to the direct summand $Z_{1/2}(\mathbf{a}) \subset T_a N$ we get a direct submanifold $B = B(n, \Lambda, R)$ of Z, and we refer to B as the *base* manifold of N. Clearly B is a holomorphic submanifold of the real analytic manifold N, and as in section 3

$$\left(\nabla_X Y\right)_a :\, = P_{1/2}(\mathbf{a}) Y'_a X_a, \qquad X, Y \in \mathfrak{D}(B), \quad a \in B,$$

is an $\mathsf{Aut}^\circ(Z)$-invariant torsionfree affine connection on B whose geodesics are the curves $\gamma(t) :\, = (\exp t\, g(\mathbf{a}, u))a$, $t \in \mathbb{R}$, for $a \in B$ and $u \in Z_{1/2}(\mathbf{a})$. Moreover, for $a \in B$ the Peirce reflection with respect to $\mathbf{a}$ is an involutory triple automorphisms of Z that fixes $\mathbf{a}$, hence it fixes $i \bigoplus_k A(e_k)$ and $Z_{1/2}(\mathbf{a})$. It is easy to see that this reflection commutes with the exponential mapping, hence it fixes $B(n, \Lambda, R)$ and os it defines a holomorphic symmetry of B. In general $(\mathbf{a})$ does not belong to B hence this symmetry in general has no fixed points in B. When the algebraic element $a \in Z$ has finite rank, that is when rank $(a) = \sum_k$ rank $(e_k) < \infty$, the subtriple $Z_{1/2}(\mathbf{a})$ is linearly equivalent to a complex Hilbert space by [12] and by using the algebraic metric of Harris one can introduce an $\mathsf{Aut}^\circ(Z)$-invariant Riemann structure and a Kähler structure on the base manifold in exactly the same way we did in section 3, and the connection ∇ turns out to be the Levi-Civita and the Kähler connection on B.

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