A NOTE ON SOME ALGEBRA CONSTRUCTIONS OVER RATIONAL FUNCTION FIELDS

S. PUMPLÜN

ABSTRACT. Let F be a field of characteristic not 2 or 3. We give easy sufficient criteria for some first Tits constructions over the rational function field $F(X)$ to yield division algebras.

Introduction

Let $F(x)$ the field of rational functions over a field F of characteristic not 2 or 3. We obtain some easy to check sufficient criteria which help to construct examples of cubic Jordan division algebras over $F(x)$ which arise as first Tits constructions out of separable cubic algebras.

In [G-R-SB], Gajivaradhan, Rema and Sri Bala gave some sufficient criteria for quaternion and octonion algebras over $F(x)$ to be division algebras, with F of characteristic unequal to 2. Their methods of proof are analogous to the ones used here.

1. Preliminaries

1.1. Let F be a field of characteristic not 2 or 3 and $\lambda \in F^\times$. Let B be a separable associative unital algebra of degree 3 over F with norm $N_{B/F}$ and trace $T_{B/F}$. We denote the first Tits construction employing B and λ by $J(B,\lambda)$. For the definition and general properties of $J(B,\lambda)$, the reader is referred to [P-R1], [McC] or [KMRT]. The norm of the Jordan algebra $J(B,\lambda)$ is given by

$$N_{J(B,\lambda)}((b_1,b_2,b_3)) = N_{B/F}(b_1) + \lambda N_{B/F}(b_2) + \lambda^2 N_{B/F}(b_3) - \lambda T_{B/F}(b_1 b_2 b_3)$$

with $b_1, b_2, b_3 \in B$. It is well-known that $J(B,\lambda)$ is a division algebra if and only if $\lambda \notin N_{B/F}(B^\times)$ if and only if $N_{J(B,\lambda)}$ is anisotropic. Moreover, $J(B,b) \cong J(B,c^3 b)$ for all $c \in F^\times$.

An *Albert algebra* over F is an exceptional simple Jordan algebra of degree 3, i.e. an F-form of the Jordan algebra of 3-by-3 hermitian matrices with diagonal entries in F and off-diagonal entries in the split octonion algebra $\mathrm{Zor}(F)$ (or details, see for instance [P-R1, 2], or [KMRT, p. 524]). Every Albert algebra over F can be obtained by a first or second Tits construction (cf. [P-R1] or [McC]).

Date: 11.10.2012.

1991 *Mathematics Subject Classification.* Primary: 17A35.

Key words and phrases. Jordan division algebras, Albert algebras, first Tits construction, rational function fields.

For an iterated first Tits construction we write $J(B,\mu,\lambda)=J(J(B,\mu),\lambda)$ or $J(B,\mu,\lambda,\alpha)=J(J(J(B,\mu),\lambda),\alpha)$ with $\mu,\lambda,\alpha\in F^\times$.

1.2. The set-up. Let $K=F(x)$ be the field of rational functions over F. A polynomial $f(x)\in F[x]$ is said to be *of the nth kind* if $f^{(i)}(0)=0$ for all $i\in\{1,\dots,n-1\}$, but $f^{(n)}(0)\neq 0$. Every element in the group $K^\times/K^{\times 3}$ is given by a polynomial of either the first, the second or the third kind.

Let B be a separable associative algebra of degree 3 over $K=F(x)$. When looking at a first Tits construction $J(B,\lambda(x))$ with $\lambda(x)\in F(x)$, $\lambda(x)=f(x)/g(x)$ with $f(x),g(x)\in F[x]$, we can 'clear the denominator' and instead look at $J(B,\widetilde{\lambda}(x))$ for a suitable $\widetilde{\lambda}(x)\in F[x]$: let $\widetilde{\lambda}=g(x)^3f(x)/g(x)=g(x)^2f(x)\in F[x]$ then $J(B,\lambda(x))\cong J(B,\widetilde{\lambda}(x))$.

So we only need to deal with the case $J(B,f(x))$, where $f(x)\in F[x]$. Let $f(x)=f_1(x)^{\varepsilon_1}\cdots f_r(x)^{\varepsilon_r}$ be the decomposition of $f(x)\in F[x]$ into distinct irreducible factors $f_1(x),\dots,f_r(x)$. Since we know that two polynomials $f(x),h(x)\in F[x]$ with $f(x)=l(x)^3h(x)$ for some $l(x)\in F[x]$ yield isomorphic Jordan algebras $J(B,f(x))\cong J(B,h(x))$, when looking at $J(B,f(x))$, we may assume without loss of generality that

$$f(x)=f_1(x)^{\varepsilon_1}\cdots f_r(x)^{\varepsilon_r}$$

with $\varepsilon_i\in\{1,2\}$ for all $i=1,\dots,r$.

Define

$$\alpha(x)\in F[x],\quad \alpha(x)=\alpha_0+\alpha_1x+\alpha_2x^2+\cdots+\alpha_tx^t,$$
$$\mu(x)\in F[x],\quad \mu(x)=\mu_0+\mu_1x+\mu_2x^2+\cdots+\mu_rx^r,$$
$$\lambda(x)\in F[x],\quad \lambda(x)=\lambda_0+\lambda_1x+\lambda_2x^2+\cdots+\lambda_sx^s,$$

Remark 1. Let $\alpha(x),\beta(x),\gamma(x)\in F[x]$ be of the first kind. Gajivaradhan, Rema and Sri Bala [G-R-SB] proved two results for octonion algebras: they showed that if the octonion algebra $\mathrm{Cay}(F,\alpha(0),\beta(0),\gamma(0))$ obtained by a repeated Cayley-Dickson doubling process out of F is a division algebra over a field F of characteristic not 2, then the octonion algebra $\mathrm{Cay}(K,\alpha(x),\beta(x),\gamma(x))$ is a division algebra over K. If $\alpha(x)$ and $\beta(x)$ are of the first kind and $\gamma(x)$ is of the second kind, and if the quaternion algebra $(\alpha(0),\beta(0))_F$ is a division algebra over F then $\mathrm{Cay}(K,\alpha(x),\beta(x),\gamma(x))$ is a division algebra over K. They proved a similar result for quaternion algebras over K. Since we know that every composition algebra over the polynomial ring $F[x]$ is defined over F [P, 6.8], we point out that for instance

$$\mathrm{Cay}(K,\alpha(x),\beta(x),\gamma(x))=\mathrm{Cay}(F[x],\alpha(x),\beta(x),\gamma(x))\otimes F(x)\cong_{F[x]}\mathrm{Cay}(F,a,b,c)\otimes_F F(x)$$

for suitable $a,b,c\in F^\times$.

2. The first Tits construction over $F(x)$ using polynomials of the first kind

Lemma 2. *Let $E=J(F(x),\alpha(x))$ with $\alpha(x)\in F[x]$ of the first kind. If*

$$E_0=J(F,\alpha_0)$$

is a division algebra over F, then E is a division algebra over $F(x)$.

Proof. Assume $E = J(F(x), \alpha(x))$ is not a division algebra over $F(x)$, then $\alpha(x) \in F(x)^{\times 3}$, which means $\alpha(x) = f(x)^3/g(x)^3$ for suitable $f(x), g(x) \in F[x]$. Since in this case $J(F(x), \alpha(x)) \cong J(F(x), f(x)^3)$, we may assume $\alpha(x) = f(x)^3$ with $f(x) \in F[x]$. This implies that $\alpha(x) = b^3 + \dots$ for some $b \in F^\times$, i.e. $\alpha_0 = b^3$ and therefore $E_0 = J(F, \alpha(0)) = J(F, \alpha_0)$ is not a division algebra, either. □

Theorem 3. *(i) Let $E = E_0 \otimes_F K$ with E_0 a separable cubic field extension over F. Let $A = J(E, \lambda(x))$ with $\lambda(x) \in F[x]$ of the first kind. If*

$$A_0 = J(E_0, \lambda_0)$$

is a division algebra over F, then A is a division algebra over $F(x)$.
(ii) Let $B = B_0 \otimes_F K$ where B_0 is a central simple associative division algebra over F. Let $J = J(B, \alpha(x))$ with $\alpha(x) \in F[x]$ of the first kind. If

$$J_0 = J(B_0, \alpha_0)$$

is an Albert division algebra over F, then J is an Albert division algebra over $F(x)$.

Proof. (i) Let $1, e, f$ be a basis of E_0 over F. Suppose that $A_0 = J(E_0, \lambda(0)) = J(E_0, \lambda_0)$ is a division algebra over F. $A = J(E, \lambda(x))$ is a division algebra over $F(x)$ if and only if $N_{A/K}$ is an anisotropic cubic form, i.e. we have to show that there are only trivial $h_i(x) \in K$ such that $0 = N_{A/K}((h_1, \dots, h_9))$. Suppose there are $h_i(x) \in K$ such that $0 = N_{A/K}((h_1, \dots, h_9))$. By clearing denominators we may assume that $h_i(x) \in F[x]$,

$$h_i = h_i(x) = \sum_{j=0}^{n_i} c_{i,j} x^j,$$

so that

$$0 = N_{A/K}((h_1, \dots, h_9)) = N_{E/K}(h_1 + h_2 e + h_3 f) + \lambda N_{E/K}(h_4 + h_5 e + h_6 f) + \lambda^2 N_{E/K}(h_7 + h_8 e + h_9 f)$$
$$-\lambda T_{E/K}((h_1 + h_2 e + h_3 f)(h_4 + h_5 e + h_6 f)(h_7 + h_8 e + h_9 f)).$$

Comparing the constants (which amounts to plugging in 0 everywhere), this yields

$$0 = N_{A_0/F}((h_1(0), \dots, h_9(0))) = N_{E_0/F}(c_{1,0} + c_{2,0} e + c_{3,0} f) + \lambda_0 N_{E_0/F}(c_{4,0} + c_{5,0} e + c_{6,0} f)$$
$$+\lambda_0^2 N_{E_0/F}(c_{7,0} + c_{8,0} e + c_{9,0} f) - \lambda_0 T_{E_0/F}((c_{1,0} + c_{2,0} e + c_{3,0} f)(c_{4,0} + c_{5,0} e + c_{6,0} f)(c_{7,0} + c_{8,0} e + c_{9,0} f)).$$

Since A_0 is division by hypothesis, this means all $c_{1,0}, \dots, c_{9,0}$ must be zero and so we have $h_i = x\widetilde{h_i}$ for all i and $N_{A_0/F}((h_1, \dots, h_9)) = x^3 N_{A_0/F}(\widetilde{h_1}, \dots, \widetilde{h_9})$. We now proceed by induction and assume that all coefficients of the h_i's up to the one of x^n are zero. Then $N_{A_0/F}((h_1, \dots, h_9)) = x^{3n} N_{A_0/F}((\widetilde{h_1}, \dots, \widetilde{h_9}))$ where now $h_i = x^n \widetilde{h_i}$ for all i and hence $0 = N_{A_0/F}((h_1, \dots, h_9))$ means $0 = N_{A_0/F}((\widetilde{h_1}, \dots, \widetilde{h_9}))$ Now compare the coefficients of the x^{n+1}'s appearing in the equation. Then by the same argument we obtain that

$$0 = N_{A_0/F}((c_{1,n+1}, \dots, c_{9,n+1})) =$$
$$N_{E_0/F}(c_{1,n+1} + c_{2,n+1} e + c_{3,n+1} f) + \lambda_0 N_{E_0/F}(c_{4,n+1} + c_{5,n+1} e + c_{6,n+1} f)$$
$$+\lambda_0^2 N_{E_0/F}(c_{7,n+1} + c_{8,0} e + c_{9,n+1} f)$$
$$-\lambda_0 T_{E_0/F}((c_{1,0} + c_{2,n+1} e + c_{3,n+1} f)(c_{4,n+1} + c_{5,n+1} e + c_{6,n+1} f)(c_{7,n+1} + c_{8,n+1} e + c_{9,n+1} f))$$

which means all $c_{1,n+1},\dots,c_{9,n+1}$ must be zero as well. By induction we thus show that $0=N_{A/K}((h_1,\dots,h_9))$ implies that $h_1=\dots=h_9=0$, hence that A is a division algebra over K.

(ii) By a well-known theorem of Wedderburn, every central simple algebra of degree 3 over F is cyclic. Suppose $B_0=(L,a)$ is a central simple division algebra of degree 3 over F, where $L=F[x]/(x^3-b)=F(z)$ is a cubic field extension of F. We give the argument for the special case that F contains a primitive cube root of unity ρ, because then the basis of the algebra is easy to write down (but the general case works analogously): B_0 has F-basis $\{l^iz^j|0\leq i,j\leq 2\}$ where

$$zl=lz\rho,\quad l^3=a\in F^\times,\quad z^3=b\in F^\times$$

[Pi, p. 299]. Suppose that $J=J(B_0,\alpha(0))=J(B_0,\alpha_0)$ is a division algebra over F. Use that

$$N_{J(B,\alpha(x))}((b_1,b_2,b_3))=N_{B/K}(b_1)+\alpha(x)N_{B/K}(b_2)+\alpha(x)^2N_{B/K}(b_3)-\alpha(x)T_{B/F}(b_1b_2b_3)$$

$J=J(B,\alpha(x))$ is a division algebra over $F(x)$ if and only if $N_{J/K}$ is an anisotropic cubic form, i.e. we have to show that there are only the trivial $h_i(x)\in K$ such that $0=N_{J/K}((h_1,\dots,h_{27}))$. Suppose there are $h_i(x)\in K$ such that $0=N_{A/K}((h_1,\dots,h_{27}))$. By clearing denominators we may assume there exist polynomials $h_i(x)\in F[x]$,

$$h_i=h_i(x)=\sum_{j=0}^{n_i}c_{i,j}x^j,$$

such that

$$0=N_{J/K}((h_1,\dots,h_{27}))=N_{B/K}(h_1+zh_2+z^2h_3+l(h_4+zh_5+z^2h_6)+l^2(h_7+zh_8+z^2h_9))+$$
$$\alpha(x)N_{B/K}(h_{10}+\dots+l^2z^2h_{18})+\alpha(x)^2N_{B/K}(h_{19}+\dots+l^2z^2h_{27})$$
$$-\alpha(x)T_{J/K}((h_1+\dots+l^2z^2h_9)(h_{10}+\dots+l^2z^2h_{18})(h_{19}+\dots+l^2z^2h_{27})).$$

The proof now works analogously as in (ii): Comparing the constants, since A_0 is division by hypothesis, all $c_{1,0},\dots,c_{27,0}$ must be zero and so we have $h_i=x\widetilde{h_i}$ for all i and $N_{A_0/F}((h_1,\dots,h_{27}))=x^3N_{A_0/F}(\widetilde{h_1},\dots,\widetilde{h_{27}})$. We now proceed by induction and assume that all coefficients of the h_i's up to the one of x^n are zero. Then $N_{A_0/F}((h_1,\dots,h_{27}))=x^{3n}N_{A_0/F}((\widetilde{h_1},\dots,\widetilde{h_27}))$ where now $h_i=x^n\widetilde{h_i}$ for all i and hence $0=N_{A_0/F}((h_1,\dots,h_{27}))$ means $0=N_{A_0/F}((\widetilde{h_1},\dots,\widetilde{h_{27}}))$ Now compare the coefficients of the x^{n+1}'s appearing in the equation. Then by the same argument we obtain that all $c_{1,n+1},\dots,c_{27,n+1}$ must be zero as well. By induction we thus show that $0=N_{A/K}((h_1,\dots,h_{27}))$ implies that $h_1=\dots=h_9=0$, hence that A is a division algebra over K. □

Remark 4. Alternatively, we can prove (i) and (ii) much quicker as follows:

(i) Identify $E_0\otimes F(x)=E_0(x)=E$. Suppose $A=J(E,\mu(x))$ is not a division algebra over $F(x)$, then $\mu(x)\in N_{E/K}(E^\times)$. This means $\mu(x)=N_{E/F(x)}(e(x))$ for a suitable non-zero $e(x)\in E_0(x)$. Substituting $x=0$, we get $\mu(0)=N_{E_0/F}(e(0))$, i.e. $\mu(0)\in N_{E_0/F}(E_0^\times)$. Therefore $A_0=J(E_0,\mu(0))=J(E_0,\mu_0)$ is not a division algebra over F.

(ii) Let $B=B_0\otimes_F K$ where B_0 is a central simple associative division algebra over F. Let $J=J(B,\alpha(x))$ with $\alpha(x)\in F[x]$ of the first kind. Suppose $A=J(B,\alpha(x))$ is not a

division algebra over $F(x)$, then $\alpha(x) \in N_{B/K}(B^\times)$. This means $\alpha(x) = N_{B/F(x)}(e(x))$ for a suitable $e(x) \in B_0 \otimes F(x)$. We get $\alpha(0) = N_{B_0/F}(e(0))$, i.e. $\alpha(0) \in N_{B_0/F}(B_0^\times)$. Therefore $A_0 = J(B_0, \alpha(0)) = J(B_0, \alpha_0)$ is not a division algebra over F.
We give the lengthy proof here as well to show the inductive nature of the argument.

Theorem 5. *(i) Let $A = J(F(x), \mu(x), \lambda(x))$ and $\mu(x), \lambda(x) \in F[x]$ of the first kind. If*

$$A_0 = J(F, \mu_0, \lambda_0)$$

is a division algebra over F, then A is a division algebra over $F(x)$.
(ii) Let $J = J(F(x), \mu(x), \lambda(x), \alpha(x))$ and $\mu(x), \lambda(x), \alpha(x) \in F[x]$ of the first kind. If

$$J_0 = J(F, \mu_0, \lambda_0, \alpha_0)$$

is a division algebra over F, then J is a division algebra over $F(x)$.

Proof. For $J = J(F(x), \mu(x))$, by plugging in zero the term $N_J(h_1(x), h_2(x), h_3(x))$ becomes $N_{J(F,\mu(0))}(h_1(0), h_2(0), h_3(0))$, i.e. the norm of $J(F, \mu(0))$ (and for $J = J(F(x), \mu(x), \lambda(x))$, $N_J(h_1(x), \dots, h_9(x))$ becomes $N_{J(F,\mu(0),\lambda(0))}(h_1(0), \dots, h_9(0))$). Hence the same induction method as in the proof of Theorem 3 can be applied, substituting N_J for N_E or N_B everywhere. □

More generally, if φ is a form of degree n over $F(x)$, we may assume without loss of generality that all its coefficients $\alpha_{i_1,\dots,i_{r_j}}(x)$ are polynomials in $F[x]$. If they are all of the first kind, the same inductive argument proves that φ is anisotropic, if the corresponding form φ_0 over F we obtain from φ by putting $\alpha_{i_1,\dots,i_{r_j}}(0)$ instead of $\alpha_{i_1,\dots,i_{r_j}}(x)$ as coefficients everywhere, is anisotropic.

3. The first Tits construction over $F(x)$ using polynomials of the second or third kind

Lemma 6. *$E = J(F(x), \alpha(x))$ is a division algebra over $F(x)$ for all $\mu(x) \in F[x]$ of the second or third kind.*

Proof. Suppose $E = J(F(x), \alpha(x))$ is not a division algebra over $F(x)$. Then $\alpha(x) \in F(x)^{\times 3}$ which means $\alpha(x) = f(x)^3/g(x)^3$ for suitable $f(x), g(x) \in F[x]$. Since $J(F(x), \alpha(x)) \cong J(F(x), f(x)^3)$ assume w.l.o.g. that $\alpha(x) = f(x)^3$ with $f(x) \in F[x]$ and $f(x) = b_0 + b_1 x + b_2 x^2 + \dots$.
Suppose $\alpha(x)$ is of the second kind, i.e. $\alpha(x) = x(\alpha_1 + \alpha_2 x + \dots) = \alpha_1 x + \alpha_2 x^2 \dots$ with $\alpha_1 \neq 0$. Comparing coefficients implies $\alpha_1 = \alpha_2 = 0$, a contradiction to our assumption that $\alpha_1 \neq 0$. Thus $\alpha(x) \notin F(x)^{\times 3}$ and $E = J(F(x), \alpha(x))$ a division algebra over $F(x)$ for every polynomial $\alpha(x)$.
Suppose $\alpha(x)$ is of the third kind, i.e. $\alpha(x) = x^2(\alpha_2 + \alpha_3 x + \dots) = \alpha_2 x^2 + \alpha_3 x^3 \dots$ with $\alpha_2 \neq 0$. Comparing coefficients again implies $\alpha_1 = \alpha_2 = 0$, a contradicting that $\alpha_2 \neq 0$. Thus $\alpha(x) \notin F(x)^{\times 3}$ and $E = J(F(x), \alpha(x))$ a division algebra over $F(x)$. □

Theorem 7. *Let E_0 be a separable cubic field extension over F, $E = E_0 \otimes_F F(x)$ defined over F and $A = J(E, \lambda(x))$ with $\lambda(x) \in F[x]$.*

(i) If $\lambda(x)$ is of the second kind then A is a division algebra over $F(x)$.
(ii) If $\lambda(x)$ is of the third kind then A is a division algebra over $F(x)$.

Proof. $A = J(E, \mu(x))$ is a division algebra over $F(x)$ if and only if $N_{A/K}$ is an anisotropic cubic form, i.e. we have to show that there are only trivial $h_i(x) \in K$ such that $0 = N_{A/K}((h_1, \dots, h_9))$. Suppose there are $h_i(x) \in K$ such that $0 = N_{A/K}((h_1, \dots, h_9))$. Clearing denominators we assume these $h_i(x) \in F[x]$,

$$h_i = h_i(x) = \sum_{j=0}^{n_i} c_{i,j} x^j,$$

such that

$$0 = N_{A/K}((h_1, \dots, h_9)) = N_{E/K}(h_1+h_2e+h_3f)+\lambda(x)N_{E/K}((h_4+h_5e+h_6f)+\lambda(x)^2N_{E/K}((h_7, h_8, h_9))$$
$$-\lambda(x)T_{E/K}((h_1 + h_2e + h_3f) \cdot (h_4 + h_5e + h_6f) \cdot (h_7 + h_8e + h_9f),$$

with $1, e, f$ a basis of E_0 over F.
(i) Let

$$\lambda(x) = \lambda_1 x + \lambda_2 x^2 + \dots + \lambda_s x^s = x(\lambda_1 + \lambda_2 x + \dots + \lambda_s x^{s-1}) = x\widetilde{\lambda}(x), \quad \lambda_1 \neq 0$$

be of the second kind. Plugging in 0 everywhere yields

$$0 = N_{E_0/F}(h_1(0) + h_2(0)e + h_3(0)f) = N_{E_0/F}(c_{1,0} + c_{2,0}e + c_{3,0}f)$$

Since E_0 is division by hypothesis, $c_{1,0} = c_{2,0} = c_{3,0} = 0$ and so we have $h_i = x\widetilde{h_i}$ for $i = 1, 2, 3$ and

$$0 = N_{A/K}((h_1, \dots, h_9))$$
$$= x^3 N_{E/F}(\widetilde{h_1} + \widetilde{h_2}e + \widetilde{h_3}f) + x\widetilde{\lambda}(x)N_{E/K}(h_4 + h_5e + h_6f) + x^2\widetilde{\lambda}^2 N_{E/K}(h_7 + h_8e + h_9f)$$
$$-x^2\widetilde{\lambda}T_{E/K}((\widetilde{h_1} + \widetilde{h_2}\alpha + \widetilde{h_3}f)(h_4 + h_5\alpha + h_6\alpha^2)(h_7 + h_8\alpha + h_9f)).$$

Cancel x:

$$0 = x^2 N_{E/F}(\widetilde{h_1} + \widetilde{h_2}e + \widetilde{h_3}f) + \widetilde{\lambda}(x)N_{E/K}(h_4 + h_5e + h_6\alpha^2) + x\widetilde{\lambda}^2 N_{E/K}(h_7 + h_8e + h_9\alpha^2)$$
$$-x\widetilde{\lambda}T_{E/K}((\widetilde{h_1} + \widetilde{h_2}e + \widetilde{h_3}f)(h_4 + h_5e + h_6f)(h_7 + h_8e + h_9f)).$$

Put $x = 0$:

$$0 = \lambda_1 N_{E_0/F}(h_4(0) + h_5(0)e + h_6(0)f) = \lambda_1 N_{E_0/F}(c_{4,0} + c_{5,0}e + c_{6,0}f).$$

Hence also $c_{4,0} = c_{5,0} = c_{6,0} = 0$ and $h_i = x\widetilde{h_i}$ for $i = 4, 5, 6$ and

$$0 = N_{A/K}((h_1, \dots, h_9))$$
$$= x^3 N_{E/F}(\widetilde{h_1} + \widetilde{h_2}e + \widetilde{h_3}\alpha^2) + x^4\widetilde{\lambda}(x)N_{E/K}(\widetilde{h_4} + \widetilde{h_5}e + \widetilde{h_6}f) + x^2\widetilde{\lambda}^2 N_{E/K}(h_7 + h_8e + h_9f)$$
$$-x^3\widetilde{\lambda}T_{E/K}((\widetilde{h_1} + \widetilde{h_2}e + \widetilde{h_3}f)(\widetilde{h_4} + \widetilde{h_5}e + \widetilde{h_6}f)(h_7 + h_8e + h_9f)).$$

Cancel x^2:

$$0 = xN_{E/F}(\widetilde{h_1} + \widetilde{h_2}e + \widetilde{h_3}f) + x^2\widetilde{\lambda}(x)N_{E/K}(\widetilde{h_4} + \widetilde{h_5}e + \widetilde{h_6}f) + \widetilde{\lambda}^2 N_{E/K}(h_7 + h_8e + h_9f)$$
$$-x\widetilde{\lambda}T_{E/K}((\widetilde{h_1} + \widetilde{h_2}e + \widetilde{h_3}f)(\widetilde{h_4} + \widetilde{h_5}e + \widetilde{h_6}f)(h_7 + h_8e + h_9f)).$$

Put $x = 0$:

$$0 = \lambda_1^2 N_{E/K}(h_7(0) + h_8(0)e + h_9(0)f).$$

Hence also $c_{7,0} = c_{8,0} = c_{9,0} = 0$ and $h_i = x\widetilde{h_i}$ for $i = 7, 8, 9$. An obvious induction now shows that we may conclude $h_1 = \cdots = h_9 = 0$ this way.
(ii) Let

$$\lambda(x) = \lambda_2 x^2 + \cdots + \lambda_s x^s = x^2(\lambda_2 + \lambda_3 x + \cdots + \lambda_s x^{s-2}) = x^2\widetilde{\lambda}(x), \quad \lambda_2 \neq 0$$

be of the third kind. Put $x = 0$, then

$$0 = N_{E_0/F}(h_1(0) + h_2(0)e + h_3(0)f) = N_{E_0/F}(c_{1,0} + c_{2,0}e + c_{3,0}f),$$

i.e. $c_{1,0} = c_{2,0} = c_{3,0} = 0$ and $h_i = x\widetilde{h_i}$ for $i = 1, 2, 3$. Now

$$\begin{aligned} 0 &= N_{A/K}((h_1, \ldots, h_9)) \\ &= x^3 N_{E/F}(\widetilde{h_1}, \widetilde{h_2}, \widetilde{h_3}) + x^2\widetilde{\lambda}(x)N_{E/K}(h_4 + h_5 e + h_6 f) + x^4\widetilde{\lambda}(x)^2 N_{E/K}(h_7 + h_8 e + h_9 f) \\ &\quad -x^3\widetilde{\lambda}(x)T_{E/K}((\widetilde{h_1} + \widetilde{h_2}e + \widetilde{h_3}f)(h_4 + h_5 e + h_6 f)(h_7 + h_8 e + h_9 f)). \end{aligned}$$

Cancel x^2:

$$\begin{aligned} 0 &= xN_{E/F}(\widetilde{h_1}, \widetilde{h_2}, \widetilde{h_3}) + \widetilde{\lambda}(x)N_{E/K}(h_4 + h_5 e + h_6 f) + x^2\widetilde{\lambda}(x)^2 N_{E/K}(h_7 + h_8 e + h_9 f) \\ &\quad -x\widetilde{\lambda}(x)T_{E/K}((\widetilde{h_1} + \widetilde{h_2}e + \widetilde{h_3}f)(h_4 + h_5 e + h_6 f)(h_7 + h_8 e + h_9 f)). \end{aligned}$$

Put $x = 0$:

$$0 = \lambda_2 N_{E_0/F}(h_4(0) + h_5(0)e + h_6(0)f) = \lambda_2 N_{E_0/F}(c_{4,0} + c_{5,0}e + c_{6,0}f).$$

Hence also $c_{4,0} = c_{5,0} = c_{6,0} = 0$ and $h_i = x\widetilde{h_i}$ for $i = 4, 5, 6$ and

$$\begin{aligned} 0 &= xN_{E/F}(\widetilde{h_1}, \widetilde{h_2}, \widetilde{h_3}) + x^3\widetilde{\lambda}(x)N_{E/K}(\widetilde{h_4} + \widetilde{h_5}e + \widetilde{h_6}f) + x^2\widetilde{\lambda}(x)^2 N_{E/K}(h_7 + h_8 e + h_9 f) \\ &\quad -x^2\widetilde{\lambda}(x)T_{E/K}((\widetilde{h_1} + \widetilde{h_2}e + \widetilde{h_3}f)(\widetilde{h_4} + \widetilde{h_5}e + \widetilde{h_6}f)(h_7 + h_8 e + h_9 f)). \end{aligned}$$

Cancel x:

$$\begin{aligned} 0 &= N_{E/F}(\widetilde{h_1}, \widetilde{h_2}, \widetilde{h_3}) + x^2\widetilde{\lambda}(x)N_{E/K}(\widetilde{h_4} + \widetilde{h_5}e + \widetilde{h_6}f) + x\widetilde{\lambda}(x)^2 N_{E/K}(h_7 + h_8 e + h_9 f) \\ &\quad -x\widetilde{\lambda}(x)T_{E/K}((\widetilde{h_1} + \widetilde{h_2}e + \widetilde{h_3}f)(\widetilde{h_4} + \widetilde{h_5}e + \widetilde{h_6}f)(h_7 + h_8 e + h_9 f)). \end{aligned}$$

Put $x = 0$:

$$0 = N_{E/F}(\widetilde{h_1}, \widetilde{h_2}, \widetilde{h_3}).$$

So here the proof differs slightly form the previous case: Hence also $c_{1,1} = c_{2,1} = c_{3,1} = 0$ and we write $\widetilde{h_i} = xf_i$ for $i = 1, 2, 3$. Then

$$\begin{aligned} 0 &= x^3 N_{E/F}(f_1 + f_2 e + f_3 f) + x^2\widetilde{\lambda}(x)N_{E/K}(\widetilde{h_4} + \widetilde{h_5}e + \widetilde{h_6}f) + x\widetilde{\lambda}(x)^2 N_{E/K}(h_7 + h_8 e + h_9 f) \\ &\quad -x^2\widetilde{\lambda}(x)T_{E/K}((f_1 + f_2 e + f_3 f)(\widetilde{h_4} + \widetilde{h_5}e + \widetilde{h_6}f)(h_7 + h_8 e + h_9 f)). \end{aligned}$$

Cancel x:

$$\begin{aligned} 0 &= x^2 N_{E/F}(f_1 + f_2 e + f_3 f) + x\widetilde{\lambda}(x)N_{E/K}(\widetilde{h_4} + \widetilde{h_5}e + \widetilde{h_6}f) + \widetilde{\lambda}(x)^2 N_{E/K}(h_7 + h_8 e + h_9 f) \\ &\quad -x\widetilde{\lambda}(x)T_{E/K}((f_1 + f_2 e + f_3 f)(\widetilde{h_4} + \widetilde{h_5}e + \widetilde{h_6}f)(h_7 + h_8 e + h_9 f)). \end{aligned}$$

Put $x = 0$:

$$0 = \lambda_2^2 N_{E/K}(h_7 + h_8 e + h_9 f)$$

Hence $c_{7,0} = c_{8,0} = c_{9,0} = 0$ and $h_i = x\widetilde{h_i}$ for $i = 7, 8, 9$. An obvious induction again shows that $h_1 = \cdots = h_9 = 0$. □

This can be generalized using the same method of proof to show:

Theorem 8. *(i) Let* $A = J(F(x), \mu(x), \lambda(x))$ *with* $\mu(x), \lambda(x) \in F[x]$, *where* $\mu(x)$ *is of the first kind such that* $J(F, \mu_0)$ *is a division algebra. If* $\lambda(x)$ *is of the second or third kind then* A *is a division algebra over* $F(x)$.
(ii) Let $J = J(F(x), \lambda(x), \mu(x), \alpha(x))$, *where* $\lambda(x), \mu(x)$ *are of the first kind and* $J(F, \lambda_0, \mu_0)$ *is a division algebra over* F. *If* $\alpha(x)$ *is of the second or third kind then* J *is a division algebra over* $F(x)$.

In particular, the above conditions are necessary in case the scalars used are monomials: e.g., given $J = J(F(x), \lambda(x), \mu(x), \alpha(x))$, if $\lambda(x) = \lambda_0, \mu(x) = \mu_0$ and $\alpha(x) = \alpha_0$ are constants (i.e., monomials of the first kind), $J = J(F(x), \lambda(x), \mu(x), \alpha(x)) = J(F, \lambda_0, \mu_0, \alpha_0) \otimes_F F(x)$, so that J is division iff so is $J(F, \lambda_0, \mu_0, \alpha_0)$, and if $\lambda(x) = \lambda_0, \mu(x) = \mu_0$ and $\alpha(x) = \alpha_1 x$ or $\alpha(x) = \alpha_2 x^2$ is of the second or third kind, J is division implies that so is $J(F, \lambda_0, \mu_0)$.

References

[G-R-SB] Gajivaradhan, P., Rema, P. S., Sri Bala, S., *Composition division algebras over a function field.* Southeast Asian Bull. Math. 33 (4) (2009), 727–731.

[KMRT] Knus, M.A., Merkurjev, A., Rost, M., Tignol, J.-P., "The Book of Involutions", AMS Coll. Publications, Vol. 44 (1998).

[McC] McCrimmon, K., "A Taste of Jordan Algebras". Universitext, Springer Verlag, New York 2004.

[P] Petersson, H. P., *Composition algebras over curves of genus zero.* Trans. AMS 337 (1) (1993), 473 – 493.

[P-R1] Petersson, H. P., Racine, M., *An elementary approach to the Serre-Rost invariant of Albert algebras.* Indag. Math., N.S., 7 (3)(1996), 343-365.

[P-R2] Petersson, H. P., Racine, M., *The Serre-Rost invariant of Albert algebras in characteristic three.* Indag. Math., N.S., 8 (4)(1997), 543-548.

[Pi] Pierce, R. S., "Associative Algebras". Springer-Verlag (Graduate Texts in Mathematics), 1982.

[Pu1] Pumplün, S., *Albert algebras over curves of genus zero and one.* J. Algebra 320 (2008), 4178 – 4214.

E-mail address: susanne.pumpluen@nottingham.ac.uk

School of Mathematical Sciences, University of Nottingham, University Park, Nottingham NG7 2RD, United Kingdom