THE ALTERNATIVE DAUGAVET PROPERTY OF C*-ALGEBRAS AND JB*-TRIPLES

Miguel Martín

Departamento de Análisis Matemático
Facultad de Ciencias
Universidad de Granada
18071 Granada, SPAIN
Email address: mmartins@ugr.es

Abstract. A Banach space X is said to have the alternative Daugavet property if for every (bounded and linear) rank-one operator $T : X \longrightarrow X$ there exists a modulus one scalar ω such that $\|Id+\omega T\| = 1+\|T\|$. We give geometric characterizations of this property in the setting of C^*-algebras, JB^*-triples and their isometric preduals.

For a real or complex Banach space X, we write X^* for its topological dual and $L(X)$ for the Banach algebra of bounded linear operators on X. We denote by $\mathbb{T}$ the set of modulus-one scalars.

Following [36], we say that a Banach space X has the *alternative Daugavet property* if every rank-one operator $T \in L(X)$ satisfies the norm identity

$$\max_{\omega\in\mathbb{T}} \|Id + \omega\, T\| = 1 + \|T\|. \tag{aDE}$$

In such a case, all weakly compact operators on X also satisfy Eq. (aDE) (see [36, Theorem 2.2]). It is clear that a Banach space X has the alternative Daugavet property whenever X^* has, but it is shown in [36, Example 4.4] that the reverse result does not hold.

Observe that Eq. (aDE) for an operator T just means that there exists a modulus-one scalar ω such that the operator $S = \omega T$ satisfies the (usual) *Daugavet equation*

$$\|Id + S\| = 1 + \|S\|. \tag{DE}$$

Therefore, the *Daugavet property* (i.e., every rank-one –equivalently, every weakly compact– operator satisfies Eq. (DE) [23, Theorem 2.3]) implies the alternative Daugavet property. Examples of spaces having the Daugavet property are $C(K)$ and $L_1(\mu)$, provided that K is perfect and μ does not have any atoms (see [43] for an elementary approach), and certain function algebras such as the disk algebra $A(\mathbb{D})$ or the algebra of bounded analytic functions H^∞ [44, 46]. The state-of-the-art on the subject can be found in [23, 45]. For very recent results we refer the reader to [4, 6, 24, 25, 39] and references therein.

Date: December 10th, 2004.
2000 *Mathematics Subject Classification.* Primary 17C65, 46B20, 46L05. Secondary 47A12.
Key words and phrases. C^*-algebra; von Neumann predual; JB^*-triple; predual of a JBW^*-triple; Daugavet equation; numerical range; strongly exposed point, extreme point.
Research partially supported by Spanish MCYT project no. BFM2003-01681 and Junta de Andalucía grant FQM-185.

Although the definition of the alternative Daugavet property is very recent, Eq. (aDE) appears explicitly in several papers from the 80's as [1, 2, 18, 19, 43], where it is proved that this equation is satisfied by all operators $T \in L(X)$ whenever $X = C(K)$ or $X = L_1(\mu)$. Actually, this result appeared in the 1970 paper [14], where Eq. (aDE) is related to a constant introduced by G. Lumer in 1968, the numerical index of a Banach space. Let us give the necessary definitions. Given an operator $T \in L(X)$, the *numerical radius* of T is

$$v(T) := \sup\{|x^*(Tx)| \ : \ x \in X,\ x^* \in X^*,\ \|x^*\| = \|x\| = x^*(x) = 1\}.$$

The *numerical index* of the space X is

$$n(X) := \max\{k \geqslant 0 \ : \ k\|T\| \leqslant v(T) \text{ for all } T \in L(X)\}.$$

It was shown in [14, pp. 483] that

an operator $T \in L(X)$ satisfies Eq. (aDE) if and only if $v(T) = \|T\|$,

thus, $n(X) = 1$ if and only if Eq. (aDE) is satisfied by all operators in $L(X)$. Therefore, the numerical index 1 implies the alternative Daugavet property. Examples of spaces with numerical index 1 are $L_1(\mu)$ and its isometric preduals, for any positive measure μ. For more information and background, we refer the reader to the monographs by F. Bonsall and J. Duncan [7, 8] and to the survey paper [33]. Recent results can be found in [26, 31, 35, 38, 40] and references therein. Let just mention that one has $v(T^*) = v(T)$ for every $T \in L(X)$, where T^* is the adjoint operator of T (see [7, §9]) and it clearly follows that $n(X^*) \leqslant n(X)$ for every Banach space X. The question if this is actually an equality seems to be open.

The alternative Daugavet property does not imply neither the Daugavet property nor numerical index 1. Indeed, *the space* $c_0 \oplus_\infty C([0,1], \ell_2)$ *has the alternative Daugavet property, but it does not have the Daugavet property and it does not have numerical index* 1 (see [36, Example 3.2] for the details).

In [36], the authors characterize the JB^*-triples having the alternative Daugavet property as those whose minimal tripotents are diagonalizing. It is also proved there that the predual of a JBW^*-triple has the alternative Daugavet property if and only if the triple does. The necessary definitions and basic results on JB^*-triples are presented in section 2.

In the present paper we give geometric characterizations of the alternative Daugavet property for JB^*-triples and for their isometric preduals. In particular, our results contain the above mentioned algebraic characterizations given in [36], but our proofs are independent. To state the main results of the paper we need to fix notation and recall some definitions.

Let X be a Banach space. The symbols B_X and S_X denote, respectively, the closed unit ball and the unit sphere of X. We write $\operatorname{co}(A)$ for the convex hull of the set A, and we will denote by $\operatorname{ext}(B)$ the set of extreme points in the convex set B. Let us fix u in S_X. We define the set $D(X,u)$ of all *states* of X relative to u by

$$D(X,u) := \{f \in B_{X^*} \ : \ f(u) = 1\},$$

which is a non-empty w^*-closed face of B_{X^*}. The norm of X is said to be *Fréchet-smooth* or *Fréchet differentiable* at $u \in S_X$ whenever there exists $\lim\limits_{\alpha \to 0} \dfrac{\|u + \alpha x\| - 1}{\alpha}$ uniformly for $x \in B_X$. A point $x \in S_X$ is said to be an *strongly exposed point* if there exists $f \in D(X,x)$ such that $\lim \|x_n - x\| = 0$ for every sequence (x_n) of elements of B_X such that $\lim \operatorname{Re} f(x_n) = 1$. If X is a dual space and f is taken from the predual, we say that x is a *w^*-strongly exposed point.* It is known that x is strongly exposed if and only if there is a point of Fréchet-smoothness in $D(X,x)$, which is actually an strongly exposing functional for x (see [12, Corollary I.1.5]). Finally, if X and Y are Banach spaces, we write $X \oplus_1 Y$ and $X \oplus_\infty Y$ to denote, respectively, the ℓ_1-sum and the ℓ_∞-sum of X and Y.

The main results of the paper are the characterizations of the alternative Daugavet property for JB^*-triples and preduals of JBW^*-triples given in Theorems 2.6 and 2.3 respectively. For a JB^*-triple X, the following are equivalent:

- (i) X has the alternative Daugavet property.
- (ii) $|x^{**}(x^*)| = 1$ for every $x^{**} \in \operatorname{ext}(B_{X^{**}})$ and every w^*-strongly exposed point x^* of B_{X^*}.
- (iii) Each elementary triple ideal of X is equal to $\mathbb{C}$.
- (iv) There exists a closed triple ideal Y with $c_0(\Gamma) \subseteq Y \subseteq \ell_\infty(\Gamma)$ for convenient index set Γ, and such that X/Y is non-atomic.
- (v) All minimal tripotents of X are diagonalizing.

For the predual X_* of a JBW^*-triple X, the following are equivalent:

- (i) X_* has the alternative Daugavet property,
- (ii) X has the alternative Daugavet property,
- (iii) $|x(x_*)| = 1$ for every $x \in \operatorname{ext}(B_X)$ and every $x_* \in \operatorname{ext}(B_{X_*})$.
- (iv) $X = \ell_\infty(\Gamma) \oplus_\infty \mathcal{N}$ for suitable index set Γ and non-atomic ideal $\mathcal{N}$.

Let us mention that geometric characterizations of the Daugavet property for C^*-algebras, JB^*-triples, and their isometric preduals, can be found in [4].

The outline of the paper is as follows. In section 2 we include some preliminary results on the alternative Daugavet property and on Banach spaces with numerical index 1. Section 3 is devoted to the above cited characterizations of the alternative Daugavet property for JB^*-triples and their isometric preduals, and we dedicate section 4 to particularize these result to C^*-algebras and von Neumann preduals, and to compile some results on C^*-algebras and von Neumann preduals with numerical index 1.

1. Preliminary results

We devote this section to summarize some results concerning the alternative Daugavet property and Banach spaces with numerical index 1, which we will use along the paper. Most of them appear implicity in [31, 36], but there are no explicit references. Therefore, we include them here for the sake of completeness.

Our starting point is McGregor's characterization of finite-dimensional spaces with numerical index 1 [32, Theorem 3.1]: *A finite-dimensional space X satisfies $n(X) = 1$ if and only if $|x^*(x)| = 1$ for all extreme points $x \in B_X$ and $x^* \in B_{X^*}$.* For infinite-dimensional X, $\operatorname{ext}(B_X)$ may be empty (e.g. c_0), so the right statement of McGregor's condition in this case should read

$$|x^{**}(x^*)| = 1 \qquad \text{for every } x^* \in \operatorname{ext}(B_{X^*}) \text{ and every } x^{**} \in \operatorname{ext}(B_{X^{**}}). \tag{1}$$

One can easily show that this condition is sufficient to ensure $n(X) = 1$. Actually, a bit more general result can be also proved easily.

Lemma 1.1. *Let X be a Banach space and let A be a subset of S_{X^*} such that $B_{X^*} = \overline{\operatorname{co}(A)}^{w^*}$. If*

$$|x^{**}(a^*)| = 1 \qquad \text{for every } a^* \in A \text{ and every } x^{**} \in \operatorname{ext}(B_{X^{**}}),$$

then $n(X) = 1$. In particular, condition (1) implies $n(X) = 1$.

Proof. Fix $T \in L(X)$. Since T^* is w^*-continuous, for every $\varepsilon > 0$ we can find $a^* \in A$ such that $\|T^*(a^*)\| \geqslant \|T\| - \varepsilon$. Now, take $x^{**} \in \operatorname{ext}(B_{X^{**}})$ with

$$|x^{**}(T^*(a^*))| = \|T^*(a^*)\|,$$

and observe that $|x^{**}(a^*)| = 1$, giving

$$v(T) = v(T^*) \geqslant |x^{**}(T^*(a^*))| \geqslant \|T\| - \varepsilon. \qquad \square$$

We do not know if condition (1) is also necessary in order to have numerical index 1, but considering strongly exposed points, in [31, Lemma 1] a (weaker) necessary condition is established. Actually, as is noted in the Remark 6 of the same paper, we can replace the hypothesis of having numerical index 1 by the hypothesis of having the alternative Daugavet property.

Lemma 1.2. *Let X be a Banach space with the alternative Daugavet property. Then:*

(a) $|x^{**}(x^*)| = 1$ *for every* $x^{**} \in \operatorname{ext}(B_{X^{**}})$ *and every w^*-strongly exposed point* $x^* \in B_{X^*}$.
(b) $|x^*(x)| = 1$ *for every* $x^* \in \operatorname{ext}(B_{X^*})$ *and every strongly exposed point* $x \in B_X$.

Recall that the dual unit ball of an Asplund space is the w^*-closed convex hull of its w^*-strongly exposed points (see [41], for instance) so, by just applying the above two lemmas, we easily get the following characterization.

Corollary 1.3. *Let X be an Asplund space. Then, the following are equivalent:*

(i) $n(X) = 1$.
(ii) *X has the alternative Daugavet property.*
(iii) $|x^{**}(x^*)| = 1$ *for every* $x^{**} \in \operatorname{ext}(B_{X^{**}})$ *and every w^*-strongly exposed point* $x^* \in B_{X^*}$.

For similar results concerning Banach spaces with the Radon-Nikodým property and numerical index 1 we refer the reader to [34].

Finally, let us mention that we do not know of any general characterization of Banach spaces having numerical index 1 which does not involve operators. In particular, we do not know if condition (1) is such a characterization.

2. JB^*-triples and preduals of JBW^*-triples

We recall that a *JB^*-triple* is a complex Banach space X with a continuous triple product $\{\cdots\} : X \times X \times X \longrightarrow X$ which is linear and symmetric in the outer variables, conjugate-linear in the middle variable, and satisfies:

(1) For all x in X, the mapping $y \longmapsto \{xxy\}$ from X to X is a hermitian operator on X and has nonnegative spectrum.
(2) The *main identity*

$$\{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\}$$

holds for all a, b, x, y, z in X.
(3) $\|\{xxx\}\| = \|x\|^3$ for every x in X.

Concerning Condition (1) above, we also recall that a bounded linear operator T on a complex Banach space X is said to be *hermitian* if $\|\exp(irT)\| = 1$ for every $r \in \mathbb{R}$. The main interest of JB^*-triples relies on the fact that, up to biholomorphic equivalence, there are no bounded symmetric domains in complex Banach spaces others than the open unit balls of JB^*-triples (see [27, 29]).

Every C^*-algebra becomes a JB^*-triple under the triple product

$$\{xyz\} := \frac{1}{2}(xy^*z + zy^*x).$$

Moreover, norm-closed subspaces of C^*-algebras closed under the above triple product form the class of JB^*-triples known as *JC^*-triples*.

By a *JBW^*-triple* we mean a JB^*-triple whose underlying Banach space is a dual space in metric sense. It is known (see [3]) that every JBW^*-triple has a unique predual up to isometric linear isomorphisms and its triple product is separately w^*-continuous in each variable. We will apply without notice that the bidual of every JB^*-triple X is a JBW^*-triple under a suitable triple product which extends the one of X [13].

Let X be a JB^*-triple. Given $a, b \in X$, we write $D(a, b)$ for the multiplication operator $x \longmapsto \{abx\}$. The elements a and b are said to be *orthogonal* if $D(a, b) = 0$ (equivalently, $D(b, a) = 0$). For $a \in X$, the conjugate linear operator $x \longmapsto \{axa\}$ is denoted by Q_a. An element $u \in X$ is said to be a *tripotent* if $\{uuu\} = u$. Associated to any tripotent u, we define the *Peirce projections*

$$P_2(u) = Q_u^2, \qquad P_1(u) = 2(D(u,u) - Q_u^2), \qquad P_0(u) = Id - 2D(u,u) + Q_u^2,$$

which are mutually orthogonal with sum Id, and ranges

$$X_j(u) := P_j(u)(X) = \left\{x \ : \ \{uux\} = \frac{j}{2}x\right\} \qquad (j = 0, 1, 2),$$

giving $X = X_2(u) \oplus X_1(u) \oplus X_0(u)$ (*Peirce decomposition*). A tripotent $u \in X$ is said to be *minimal* if $u \neq 0$ and $X_2(u) = \mathbb{C}u$ (equivalently, $\{uXu\} = \mathbb{C}u$) and u is said to be *diagonalizing* if $X_1(u) = 0$. A *triple ideal* of X is a subspace M of X such that $\{XXM\} + \{XMX\} \subseteq M$. It is well-known that the triple ideals of a JB^*-triple are precisely its M-ideals [3, Theorem 3.2]. The *socle* of X, $K_0(X)$, is the closed linear span of the minimal tripotents of X. Then, $K_0(X)$ is a triple ideal of X (the one generated by the minimal tripotents in X), which is equals to the c_0-sum of every elementary triple ideal of X [10, Lemma 3.3]. X is said to be *non-atomic* if $K_0(X) = 0$, that is, if it does not contain any minimal tripotent. By [36, Theorem 4.7] or [4, Corollary 3.4], a JB^*-triple is non-atomic if and only if it has the Daugavet property. On the other side, a JBW^*-tripe Y is said to be *atomic* if it equals the w^*-closure of it socle. The JBW^*-triple Y is said to be a *factor* if it cannot be written as an ℓ_∞-sum of two nonzero ideals, or equivalently [21] if 0 and Y are the only w^*-closed ideals in Y. An specially important class of JBW^*-triples are the so-called *Cartan factors*. This class falls into six subclasses as follows, where H and H' are arbitrary complex Hilbert spaces and $J : H \longrightarrow H$ is a conjugation (that is, a conjugate linear isometry with $J^2 = Id$):

(1) *rectangular:* $L(H, H')$;
(2) *symplectic:* $\{x \in L(H) \ : \ Jx^*J = -x\}$;
(3) *hermitian:* $\{x \in L(H) \ : \ Jx^*J = x\}$;
(4) *spin:* H with $\dim(H) \geqslant 3$, with convenient (non-Hilbertian) equivalent norm.
(5) $M_{1,2}(\mathbb{O})$: the 1×2 matrices over the complex octonians $\mathbb{O}$;
(6) $H_3(\mathbb{O})$: the hermitian 3×3 matrices over $\mathbb{O}$.

The rectangular, symplectic and hermitian factors have the usual JC^*-triple product. For the definition of the other triple products see [29]. The Cartan factors are appropriately "irreducible" JBW^*-triples [20] in terms of which there is a useful representation theory. For instance, every JB^*-triple is isometrically isomorphic to a subtriple of an ℓ_∞-sum of Cartan factors [17].

A JB^*-triple M is said to be *elementary* if it is isometric to the socle $K_0(C)$ of a Cartan factor C. We have $K_0(C)^{**} = C$, and that a JB^*-triple is elementary if and only if its bidual is a Cartan factor [9, Lemma 3.2]. Of course, no elementary JB^*-tripe has the Daugavet property. The following result said that, actually, not so many elementary JB^*-triples have the alternative Daugavet property.

Proposition 2.1. *The only Cartan factor which has the alternative Daugavet property is $\mathbb{C}$. Actually, the only elementary JB^*-triple which has the alternative Daugavet property is $\mathbb{C}$.*

To prove the above proposition, we need the following easy consequence of [16, Proposition 4] and [5, Corollary 2.11] (see the proof of [4, Theorem 3.2]). For the sake of completeness, we include a proof here.

Lemma 2.2. *Let X be a JBW^*-triple. Then, every extreme point x_* of B_{X_*} is strongly exposed by a minimal tripotent of X.*

Proof. Given $x_* \in \operatorname{ext}(B_{X_*})$, [16, Proposition 4] assures the existence of a minimal tripotent u of X such that $u(x_*) = 1$. But u is a point of Fréchet-smoothness of the norm of X [5, Corollary 2.11] so, as we commented in the introduction, this implies that x_* is strongly exposed by u (see [12, Corollary I.1.5], for instance). □

Proof of Proposition 2.1. Let M be an elementary JB^*-triple and write $C = M^{**}$, which is a Cartan factor. Suppose that M has the alternative Daugavet property. By the above lemma, every extreme point x^* of B_{M^*} is strongly exposed by a minimal tripotent of M^{**}, which is actually in M, so x^* is a w^*-strongly exposed point of B_{M^*}. Then, Lemma 1.2 gives us that

$$|e(x_*)| = 1 \qquad \big(e \in \operatorname{ext}(B_C)\,,\ x_* \in \operatorname{ext}(B_{C_*})\big). \tag{2}$$

We claim that

$$|u(x_*)| \in \{0, 1\} \tag{3}$$

for every minimal tripotent u of C and every $x_* \in \operatorname{ext}(B_{C_*})$. Indeed, we fix a minimal tripotent u of C and $x_* \in \operatorname{ext}(B_{C_*})$, and we find a maximal orthogonal family of minimal tripotents $\{u_i\}_{i\in I}$ such that $u_{i_0} = u$ for some $i_0 \in I$. Now, for each family $\{\lambda_i\}_{i\in I}$ of elements of $\mathbb{T}$, we consider $e = \sum_{i\in I} \lambda_i u_i$, which is a complete tripotent of C [42, Lemma 2.1] and thus, an extreme point of B_C [30], so Eq. (2) gives

$$\left|\sum_{i\in I} \lambda_i u_i(x_*)\right| = |e(x_*)| = 1.$$

In particular, choosing a suitable family $\{\lambda_i\}_{i\in I}$, we get

$$\sum_{i\in I} |u_i(x_*)| = 1.$$

If we write $\alpha = u_{i_0}(x_*)$, $\beta = \sum_{i\in I\setminus\{i_0\}} u_i(x_*)$, the above two equations give that

$$|\alpha| \leqslant 1,\ |\beta| \leqslant 1, \qquad \text{and} \qquad |\alpha + \lambda\beta| = 1 \ \text{ for every } \lambda \in \mathbb{T}.$$

This implies that $|\alpha|, |\beta| \in \{0, 1\}$, and the claim is proved.

Now, the set of minimal tripotents of a Cartan factor is connected; indeed, for Cartan factors of finite-rank this can be easily deduced from [28, (4.5) and (4.6)], and the remaining examples follows by a direct inspection (compare [28, Section 3]). Therefore, Eq. (3) gives us that $|u(x_*)| = 1$ for every minimal tripotent u of C and every $x_* \in \operatorname{ext}(B_{C_*})$. This implies that every minimal tripotent of C is complete (there is no pair of orthogonal minimal tripotents in C) and thus, $\operatorname{rank}(C) = 1$ and C is a Hilbert space [11]; but, obviously, if a Hilbert space satisfies (2), then it is one-dimensional. □

If X is a JBW^*-triple, it is well known that $X_* = A \oplus_1 N$, where A is the closed linear span of the extreme points of B_{X_*}, and the unit ball of N has no extreme points [16]. Therefore, $X = \mathcal{A} \oplus_\infty \mathcal{N}$, where $\mathcal{A} = N^\perp \equiv A^*$ is atomic and $\mathcal{N} = A^\perp \equiv N^*$ is non-atomic. On one hand, by [17, Proposition 2.2] and [20, Corollary 1.8], an atomic JBW^*-triple is the ℓ_∞-sum of Cartan factors. On the other hand, the non-atomic part $\mathcal{N}$ of X has the Daugavet property, hence the alternative Daugavet property. Now, since an ℓ_∞-sum of Banach spaces has the alternative Daugavet property if and only if all the summands do [36, Proposition 3.1], the above proposition said that X has the alternative Daugavet property if and only if all the Cartan factors appearing in the atomic part are equal to $\mathbb{C}$. This is part of the following result, which characterizes those JBW^*-triples whose preduals have the alternative Daugavet property.

Theorem 2.3. *Let X be a JBW^*-triple. Then, the following are equivalent:*

(i) *X has the alternative Daugavet property.*
(ii) *X_* has the alternative Daugavet property.*
(iii) *$|x(x_*)| = 1$ for every $x \in \operatorname{ext}(B_X)$ and every $x_* \in \operatorname{ext}(B_{X_*})$.*
(iv) *The atomic part of X is isometrically isomorphic to $\ell_\infty(\Gamma)$ for convenient set Γ; i.e., $X = \ell_\infty(\Gamma) \oplus_\infty \mathcal{N}$ where $\mathcal{N}$ is non-atomic.*

Proof. $(i) \Rightarrow (ii)$ is clear.

$(ii) \Rightarrow (iii)$. Just apply Lemmas 1.2 and 2.2.

$(iii) \Rightarrow (iv)$. By the comments preceding this theorem, we only have to prove that every Cartan factor appearing in the atomic decomposition of X is equal to $\mathbb{C}$. Let us fix such a Cartan factor C, and let K be an elementary JB^*-triple with $K^{**} = C$. Since K^* is an L-summand of X_* and K^{**} is an M-summand of X, it is straightforward to show that condition (iii) implies

$$|x^{**}(x^*)| = 1 \qquad \big(x^{**} \in \operatorname{ext}(B_{K^{**}}),\ x^* \in \operatorname{ext}(B_{K^*})\big).$$

Now, Lemma 1.1 implies that $n(K) = 1$, so K has the alternative Daugavet property and Proposition 2.1 gives us that K (and so does C) is equal to $\mathbb{C}$.

$(iv) \Rightarrow (i)$. Since $n\big(\ell_\infty(\Gamma)\big) = 1$ and $\mathcal{N}$ has the Daugavet property, both have the alternative Daugavet property and so does its ℓ_∞-sum [36, Proposition 3.1]. □

The equivalences $(i) \Leftrightarrow (ii) \Leftrightarrow (iv)$ of the above theorem appear in [36, Theorem 4.6 and Corollary 4.8] with a different proof.

Just applying Theorem 2.3 to the JBW^*-triple X^{**}, we get the following result.

Corollary 2.4. *Let X be a JB^*-triple. If X^* has the alternative Daugavet property, then $|x^{**}(x^*)| = 1$ for every $x^{**} \in \operatorname{ext}(B_{X^{**}})$ and every $x^* \in \operatorname{ext}(B_{X^*})$. Thus, $n(X) = 1$.*

Remark 2.5. It is worth mentioning that, for an arbitrary Banach space Z, the condition

$$|z^*(z)| = 1 \qquad \text{for every } z \in \operatorname{ext}(B_Z) \text{ and every } z^* \in \operatorname{ext}(B_{Z^*})$$

does not necessarily imply that Z has the alternative Daugavet property. Indeed, let H be the 2-dimensional Hilbert space and let us consider $Z = c_0(H)$. Since

$\operatorname{ext}(B_Z) = \emptyset$, it is clear that Z satisfies the above condition. But since Z is an Asplund space and $n(Z) = n(H) < 1$ by [37, Proposition 1], Corollary 1.3 gives us that Z does not have the alternative Daugavet property.

The last result of the section is a characterization of the alternative Daugavet property for JB^*-triples.

Theorem 2.6. *Let X be a JB^*-triple. Then, the following are equivalent:*

(i) *X has the alternative Daugavet property.*
(ii) *$|x^{**}(x^*)| = 1$ for every $x^{**} \in \operatorname{ext}(B_{X^{**}})$ and every w^*-strongly exposed point x^* of B_{X^*}.*
(iii) *Each elementary triple ideal of X is equal to $\mathbb{C}$; equivalently, $K_0(X)$ is isometric to $c_0(\Gamma)$ for some index set Γ.*
(iv) *There exists a closed triple ideal Y with $c_0(\Gamma) \subseteq Y \subseteq \ell_\infty(\Gamma)$ for convenient index set Γ, such that X/Y is non-atomic.*
(v) *All minimal tripotents of X are diagonalizing.*

Proof. *(i)* $\Rightarrow$ *(ii)*. This is the first part of Lemma 1.2.

(ii) $\Rightarrow$ *(iii)*. Let K be an elementary triple ideal of X. Since K is an M-ideal of X, it is clear that condition *(ii)* goes down to K. Since K is an Asplund space, Corollary 1.3 gives us that K has the alternative Daugavet property, so $K = \mathbb{C}$ by Proposition 2.1.

(iii) $\Rightarrow$ *(iv)*. The JBW^*-triple X^{**} decomposed into its atomic and non-atomic part as $X^{**} = \mathcal{A} \oplus_\infty \mathcal{N}$. Let us consider $Y = \mathcal{A} \cap X$. By [9, Proposition 3.7], Y is an Asplund space and X/Y is non-atomic. Since $K_0(Y) = K_0(X)$ (see [10, Corollary 3.5]) and $K_0(X) = c_0(\Gamma)$, Proposition 4.4 of [10] shows that

$$c_0(\Gamma) \subset Y \subset c_0(\Gamma)^{**} = \ell_\infty(\Gamma).$$

(iv) $\Rightarrow$ *(i)*. On one hand, X/Y has the alternative Daugavet property since it has the Daugavet property. On the other hand, a sight to [15, Remark 8] gives us that $n(Y) = 1$ and thus, Y has the alternative Daugavet property. Since Y is an M-ideal, Proposition 3.4 of [36] gives us that X has the alternative Daugavet property.

(iii) $\Leftrightarrow$ *(v)*. On one hand, a minimal tripotent of X is diagonalizing if and only if it is diagonalizing in the unique Cartan factor of X^{**} containing it. On the other hand, a Cartan factor has a diagonalizing minimal tripotent (if and) only if it equal to $\mathbb{C}$; indeed, if a Cartan factor C has a minimal diagonalizing tripotent u, then $C = C_0(u) \oplus C_2(u) = C_0(u) \oplus \mathbb{C}u$; but the above direct sum is an ℓ_∞-sum (see [16, Lemma 1.3]) so, by definition of factor, $C_0(u) = 0$. □

The equivalence *(i)* $\Leftrightarrow$ *(v)* of the above theorem appear in [36, Theorem 4.5] with a different proof..

Remark 2.7. It is worth mentioning that condition *(ii)* of Theorem 2.6 does not necessarily imply the alternative Daugavet property. For instance, let H be the two-dimensional Hilbert space and let us consider $X = \ell_1(H)$. Since X has the Radon-Nikodým property and $n(X) = n(\ell_1(H)) < 1$ by [37, Proposition 1], X does not have the alternative Daugavet property (see [37, Remark 6]). But since the norm of X is not Fréchet-smooth at any point, the unit ball of X^* does not have any w^*-strongly exposed point, thus condition *(ii)* of Theorem 2.6 is satisfied.

3. C^*-algebras and von Neumann preduals

The last section of the paper is devoted to particularize the results of the above section to C^*-algebras and von Neumann preduals. Let us introduce the basic definitions and some preliminary results.

If X is a C^*-algebra, we write $Z(X)$ for the center of the algebra, i.e. the subalgebra consisting in those element of X which commutes with all elements of X. A *projection* in X is an element $p \in X$ such that $p^* = p$ and $p^2 = p$. The tripotents of X are the *partial isometries*, i.e., elements $u \in X$ satisfying that $uu^*u = u$. It is clear that projections are partial isometries (and so tripotents), but there are partial isometries which are not projections. A projection p in X is said to be *atomic* if $p \neq 0$ and $pXp = \mathbb{C}p$. The C^*-algebra X is said to be *non-atomic* if it does not have any atomic projection. It is easy to show (see [36, pp. 170] or the paragraph before [4, Corollary 4.4], for example) that a partial isometry u in X is minimal if and only if $d = u^*u$ and $r = uu^*$ (the domain and range projections associated to u) are atomic. Therefore, a C^*-algebra is non-atomic (as an algebra) if and only if it does not have any minimal tripotent, i.e., it is non-atomic as a JB^*-triple.

The following two corollaries particularize the main results of the paper to the case of C^*-algebras. The first one is a direct consequence of Theorem 2.3; the second one follows from Theorem 2.6 with just a little bit of work.

Corollary 3.1. *Let X be a von Neumann algebra. Then, the following are equivalent:*

(i) *X has the alternative Daugavet property.*
(ii) *X_* has the alternative Daugavet property.*
(iii) *$|x(x_*)| = 1$ for every $x \in \operatorname{ext}(B_X)$ and every $x_* \in \operatorname{ext}(B_{X_*})$.*
(iv) *$X = \mathcal{C} \oplus_\infty \mathcal{N}$ where $\mathcal{C}$ is commutative and $\mathcal{N}$ is non-atomic.*

Corollary 3.2. *Let X be a C^*-algebra. Then, the following are equivalent:*

(i) *X has the alternative Daugavet property.*
(ii) *$|x^{**}(x^*)| = 1$ for every $x^{**} \in \operatorname{ext}(B_{X^{**}})$ and every w^*-strongly exposed point x^* of B_{X^*}.*
(iii) *There exists a two-side commutative ideal Y such that X/Y is non-atomic.*
(iv) *$K_0(X)$ is isometric to $c_0(\Gamma)$.*
(v) *$K_0(X)$ is commutative.*
(vi) *All atomic projections in X are central.*
(vii) *$K_0(X) \subseteq Z(X)$.*

Proof. *(i)* $\Leftrightarrow$ *(ii)* $\Leftrightarrow$ *(iii)* $\Leftrightarrow$ *(iv)* follows from Theorem 2.6.

(iv) $\Leftrightarrow$ *(v)*. We have that $K_0(X) = \left[\oplus_{\gamma\in\Gamma} K(H_\gamma)\right]_{c_0}$ for convenient index set Γ and convenient family of Hilbert spaces $\{H_\gamma\}$. Thus, $K_0(X)$ is commutative if and only if $\dim(H_\gamma) \leqslant 1$ for every $\gamma \in \Gamma$.

(v) $\Rightarrow$ *(vi)*. First, we observe that every atomic projection of X is contained in $K_0(X)$. Now, we take an atomic projection $p \in X$ and we claim that $p \in Z(X)$. Indeed, fix $x \in X$ and observe that $p, px, xp \in K_0(X)$ (since it is an ideal of X), so

$$px = p(px) = (px)p = p(xp) = (xp)p = xp.$$

(vi) $\Rightarrow$ *(vii)*. We are going to show that every minimal partial isometry on X is central, which implies $K_0(X) \subseteq Z(X)$. Indeed, let $u \in X$ be a minimal partial isometry. We write $d = u^*u$ and $r = uu^*$ for the domain and range projections of

u, which are atomic and so, $d, r \in Z(X)$. Then

$$r = r^2 = (uu^*)(uu^*) = u\big((u^*u)u^*\big) = u\big(u^*(u^*u)\big) = u(u^*u^*)u = \lambda u,$$

where the last equality comes from the fact that u is minimal. Since $r \neq 0$ (otherwise $u = ru = 0$), $u \in \operatorname{span}(r) \subseteq Z(X)$.

(vii) $\Rightarrow$ *(v)* is immediate. □

We finish the paper by compiling some known results concerning C^*-algebras and von Neumann preduals having numerical index 1. Since we have not found explicitly some of these results in the literature, and we can now relate them to the alternative Daugavet property, we include them here for the sake of completeness.

C^*-algebras with numerical index 1 have been characterized in [22] (see also [26, pp. 202]) as the commutative ones. With not much work, we can obtain the following result.

Proposition 3.3. *Let X be a C^*-algebra. Then, the following are equivalent:*

(i) $n(X^*) = 1$.
(ii) *X^* has the alternative Daugavet property.*
(iii) $|x^{**}(x^*)| = 1$ *for every* $x^{**} \in \operatorname{ext}(B_{X^{**}})$ *and every* $x^* \in \operatorname{ext}(B_{X^*})$.
(iv) $n(X) = 1$.
(v) *X is commutative.*

Proof. *(i)* $\Rightarrow$ *(ii)* is clear. *(ii)* $\Rightarrow$ *(iii)* is Corollary 2.4. *(iii)* $\Rightarrow$ *(iv)* is Lemma 1.1. *(iv)* $\Rightarrow$ *(v)* is the already mentioned result of [22]. Finally, *(v)* $\Rightarrow$ *(i)* follows from the fact that X^{**} is also commutative and so, $n(X^*) \geqslant n(X^{**}) = 1$. □

Remark 3.4. In [36, Example 4.4] it is shown an example of a Banach spaces with the alternative Daugavet property whose dual does not share this property. The above proposition gives a lot of examples of this kind: *every non-atomic and non-commutative C^*-algebra.* Actually, the example given there fits this scheme.

For von Neumann preduals, we have the following.

Proposition 3.5. *Let X be a von Neumann algebra. Then, the following are equivalent:*

(i) $n(X) = 1$.
(ii) $|x^*(x)| = 1$ *for every* $x^* \in \operatorname{ext}(B_{X^*})$ *and every* $x \in \operatorname{ext}(B_X)$.
(iii) $n(X_*) = 1$.

Proof. *(i)* $\Rightarrow$ *(ii)* is immediate, since X is commutative and hence, a $C(K)$ space.

(ii) $\Rightarrow$ *(iii)* follows from Lemma 1.1 applied to X_*.

(iii) $\Rightarrow$ *(i)* is done in [26, Proposition 1.4]. □

Let us finish the paper by raising the following question: *is it true that, for a Banach space Y, $n(Y^*) = 1$ whenever $n(Y) = 1$?* We have seen (Propositions 3.3 and 3.5 above) that this is the case when either Y is a C^*-algebra or Y is the predual of a von Neumann algebra. We do not know the answer even in the case when Y is a JB^*-triple or when Y is the predual of a JBW^*-triple.

Acknowledgment: The author would like to express his gratitude to Ángel Rodríguez Palacios for his valuable suggestions and to Antonio M. Peralta and Armando R. Villena for fruitful conversations about the subject of this paper.

References

[1] Y. Abramovich, A generalization of a theorem of J. Holub, *Proc. Amer. Math. Soc.* **108** (1990), 937–939.

[2] Y. Abramovich, New classes of spaces on which compact operators satisfy the Daugavet equation, *J. Operator Theory* **25** (1991), 331–345.

[3] T. Barton and R. Timoney, Weak*-continuity of Jordan triple products and applications, *Math. Scand.* **59** (1986), 177–191.

[4] J. Becerra Guerrero, and M. Martín, The Daugavet property of C^*-algebras, JB^*-triples, and of their isometric preduals, *J. Funct. Anal.* (to appear). Available at `http://arXiv.org/abs/math.FA/0407214`.

[5] J. Becerra and A. Rodriguez-Palacios, Strong subdifferentibility of the norm on JB^*-triples, *Quartely J. Math.* **54** (2003), 381–390.

[6] D. Bilik, V. Kadets, R. Shvidkoy, and D. Werner, Narrow operators and the Daugavet property for ultraproduts, *Positivity* (to appear).

[7] F. Bonsall and J. Duncan, *Numerical Ranges of operators on normed spaces and of elements of normed algebras*, London Math. Soc. Lecture Note Ser. **2**, Cambridge, 1971.

[8] F. Bonsall and J. Duncan, *Numerical Ranges II*, London Math. Soc. Lecture Note Ser. **10**, Cambridge, 1973.

[9] L. Bunce and C.-H. Chu, Dual spaces of JB^*-triples and the Radon-Nikodym property, *Math. Z.* **208** (1991), 327–334.

[10] L. Bunce and C.-H. Chu, Dual spaces of JB^*-triples and the Radon-Nikodym property, *Math. Z.* **208** (1991), 327–334.

[11] T. Dang and Y. Friedman, Classification of JBW^*-triple factors and applications, *Math. Scand.* **61** (1987), 292–330.

[12] R. Deville, G. Godefroy, and V. Zizler, *Smoothness and Renormings in Banach spaces*, Pitman Monographs **64**, New York 1993.

[13] S. Dineen, The second dual of a JB^*-triple system, in: *Complex Analysis, Functional Analysis and Approximation Theory* (ed. by J. Múgica), 67–69, North-Holland Math. Stud. **125**, North-Holland, Amsterdam-New York, 1986.

[14] J. Duncan, C. McGregor, J. Pryce, and A. White, The numerical index of a normed space, *J. London Math. Soc.* **2** (1970), 481–488.

[15] C. Finet, M. Martín, and R. Payá, Numerical index and renorming, *Proc. Amer. Math. Soc.* **131** (2003), 871–877.

[16] Y. Friedman and B. Russo, Structure of the predual of a JBW^*-triple, *J. Reine Angew. Math.* **356** (1985), 67–89.

[17] Y. Friedman and B. Russo, The Gelfand-Naimark Theorem for JB^*-triples, *Duke Math. J.* **53** (1986), 139–148.

[18] J. Holub, A property of weakly compact operators on $C[0,1]$, *Proc. Amer. Math. Soc.* **97** (1986), 396–398.

[19] J. Holub, Daugavet's equation and operators on $L_1(\mu)$, *Proc. Amer. Math. Soc.* **100** (1987), 295–300.

[20] G. Horn, Classification of JBW^*-triples of type I, *Math. Z.* **196** (1987), 271–291.

[21] G. Horn, Characterization of the predual and ideal structure of a JBW^*-triple, *Math. Scand.* **61** (1987), 117–133.

[22] T. Huruya, The normed space numerical index of C^*-algebras, *Proc. Amer. Math. Soc.* **63** (1977), 289–290.

[23] V. Kadets, R. Shvidkoy, G. Sirotkin, and D. Werner, Banach spaces with the Daugavet property, *Trans. Amer. Math. Soc.* **352** (2000), 855–873.

[24] V. Kadets, N. Kalton, and D. Werner, Remarks on rich subspaces of Banach spaces, *Studia Math.* **159** (2003), 195–206.

[25] V. Kadets and D. Werner, A Banach space with the Schur and the Daugavet property, *Proc. Amer. Math. Soc.* **132** (2004), 1765–1773.

[26] A. Kaidi, A. Morales, and A. Rodríguez-Palacios, Geometrical properties of the product of a C^*-algebra, *Rocky Mountain J. Math.* **31** (2001), 197–213.

[27] W. Kaup, Algebraic characterization of symmetric complex Banach manifolds, *Math. Ann.* **228** (1977), 39–64.

[28] W. Kaup, Über dier Klassifikation der symmetrischen hermiteschen Mannigfaltigkeiten unendlicher Dimension. I, *Math. Ann.* **257** (1981), 463–486.

[29] W. Kaup, A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces, *Math. Z.* **183** (1983), 503–529.

[30] W. Kaup and H. Upmeier, Jordan algebras and symmetric Siegel domains in Banach spaces, *Math. Z.* **157** (1977), 179–200.

[31] G. López, M. Martín, and R. Payá, Real Banach spaces with numerical index 1, *Bull. London Math. Soc.* **31** (1999), 207–212.

[32] C. McGregor, Finite dimensional normed linear spaces with numerical index 1, *J. London. Math. Soc.* **3** (1971), 717–721.

[33] M. Martín, A survey on the numerical index of a Banach space, *Extracta Math.* **15** (2000), 265–276.

[34] M. Martín, Banach spaces having the Radon-Nikodým property and numerical index 1, *Proc. Amer. Math. Soc.* **131** (2003), 3407–3410.

[35] M. Martín, J. Merí, and A. Rodríguez-Palacios, Finite-dimensional Banach spaces with numerical index zero, *Indiana U. Math. J.* (to appear).

[36] M. Martín and T. Oikhberg, An alternative Daugavet property, *J. Math. Anal. Appl.* **294** (2004), 158–180.

[37] M. Martín and R. Payá, Numerical index of vector-valued function spaces, *Studia Math.* **142** (2000), 269–280.

[38] M. Martín and A. Villena, Numerical index and Daugavet property for $L_\infty(\mu, X)$, *Proc. Edinburgh Math. Soc.* **46** (2003), 415–420.

[39] T. Oikhberg, The Daugavet property of C^*-algebras and non-commutative L_p-spaces, *Positivity* **6** (2002), 59–73.

[40] T. Oikhberg, Spaces of operators, the ψ-Daugavet property, and numerical indices, *Positivity* (to appear).

[41] R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, Lecture Notes in Math. **1364**, Springer-Verlag, Berlin 1993.

[42] B. Sheppard, A Stone-Weierstrass theorem for JB^*-triples, *J. London Math. Soc.* **65** (2002), 381–396.

[43] D. Werner, An elementary approach to the Daugavet equation, in: *Interaction between Functional Analysis, Harmonic Analysis and Probability* (N. Kalton, E. Saab and S. Montgomery-Smith editors), Lecture Notes in Pure and Appl. Math. **175** (1994), 449–454.

[44] D. Werner, The Daugavet equation for operators on function spaces, *J. Funct. Anal.* **143** (1997), 117–128.

[45] D. Werner, Recent progress on the Daugavet property, *Irish Math. Soc. Bull.* **46** (2001), 77-97.

[46] P. Wojtaszczyk, Some remarks on the Daugavet equation, *Proc. Amer. Math. Soc.* **115** (1992), 1047–1052.