Structure and representations of Jordan algebras arising from intermolecular recombination

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May 25, 2007

Abstract

In this work we investigate the structure and representations of Jordan algebras arising from intermolecular recombination. It is proved that the variety of all these algebras is special. The basis and multiplication table are built for the free algebra of this variety. It is also shown that all the identities satisfying the operation of intermolecular recombination are consequences of only one identity of degree 4.

Introduction

The commutative algebra J over the field F is the algebra arising from intermolecular recombination if it satisfies the identity

$$\left(x^2 \cdot y\right) \cdot z + 2((x \cdot z) \cdot x) \cdot y - 2(x^2 \cdot z) \cdot y = x^2 \cdot (zy) . \tag{1}$$

Assuming that $z = x$ in the identity (1), we get the identity $(x^2 \cdot y) \cdot x = x^2 \cdot (x \cdot y)$. Consequently, all the algebras arising from intermolecular recombination are Jordan algebras. We will denote by IR-algebras[1] the algebras arising from intermolecular recombination. Let IR denote the variety of all IR-algebras.

The IR-algebras were introduced by M. Bremner in the work [1] and naturally formalized IR-operations – operations of intermolecular recombination.

In the general theory of DNA computing (see G. Păun, G. Rozenberg and A. Salomaa [2], L. Landweber and L. Kari [3]) the IR-operation has the form

[1] IR - intermolecular recombination

$$u_1xv_1 + u_2xv_2 \Rightarrow u_1xv_2 + u_2xv_1,$$

where u_1, u_1, v_1, v_2, x are words over some alphabet S. Formalization of IR-operations by M. Bremner [1] can be defined as follows. Let consider the free F-module J generated by the set $C = A \times B$, where $A = \{a_i, i \in I_1\}$, $B = \{b_i, i \in I_2\}$ - some finite or countable sets. Let turn F-module J into F-algebra by defining the operation "$\triangleright$" (splicing) on basis elements a_ib_j ($i \in I_1, j \in I_2$) following the rule

$$a_ib_j \triangleright a_kb_l = a_ib_l,$$

and extend it linearly to all of the F-module J. It is easy to check that this operation is associative. M.Bremner notes that the operation of intermolecular recombination

$$a_ib_j \cdot a_kb_l = \frac{1}{2}(a_ib_l + a_kb_j)$$

is a symmetrized product on the algebra $(J, \triangleright, +)$. And in fact,

$$a_ib_j \circ a_kb_l = \frac{1}{2}(a_ib_j \triangleright a_kb_l + a_kb_l \triangleright a_ib_j) = \frac{1}{2}(a_ib_l + a_kb_j) = a_ib_j \cdot a_kb_l.$$

So, the algebra J of intermolecular recombination is a special Jordan algebra, i.e.

$$(J, \circ, +) = (J, \triangleright, +)^{(+)}.$$

In the work [1] it is proved that the algebra J satisfies the identity (1) and all the identities of degree ≤ 6 of this algebra are consequences of the identity (1). In the same work we get the question if all the IR-algebras are special Jordan algebras.

In this work we investigate the structure and representations of IR-algebras. We prove that all the identities of the algebra J are consequences of the identity (1) and that the variety of IR-algebras is special.

1.1. Standard IR-algebras. Definitions and notations

All algebras in this work are considered over the field F of characteristic 0, so the defining identities of varieties are linearized. We will use right-handed bracketing in non-associative words. Standard definitions and notations can be found in [4].

Let basis elements a_ib_j of the F-module generated by the set $C = A \times B$ be a_{ij} where $i, j \in N$. Then the associative splicing operation defines the associative algebra C with the basis a_{ij}, $i, j \in N$ and the following multiplication table:

$$a_{ij}a_{kl} = a_{il}. \tag{2}$$

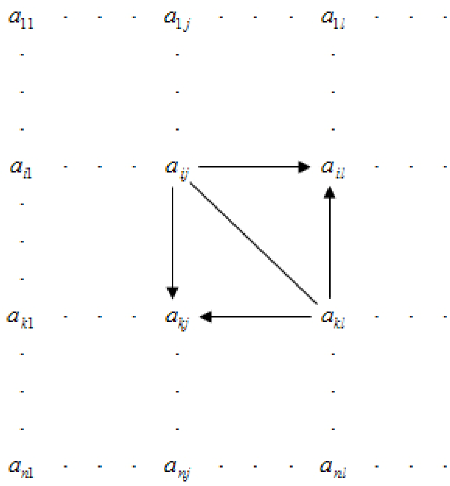
In cases when $|A| = |B| = \infty$; $|A| = n, |B| = m$; $|A| = n, |B| = \infty$, the corresponding associative algebras with the multiplication table (2) will be denoted correspondingly C_∞; C_{nm}; C_n.

We will call C_∞; C_{nm}; C_n the standard algebras of splicing, or standard S-algebras for short. We'll also define the Jordan algebras $J_\infty = C_\infty^{(+)}$, $J_{n,m} = C_{n,m}^{(+)}$, $J_n = C_n^{(+)}$ and call them standard algebras of intermolecular recombination, or standard IR-algebras for short.

It is clear that the standard IR-algebras have the basis a_{ij}, $i, j \in N$ ($1 \le i \le n$, $1 \le j \le m$ for $J_{n,m}$ and $1 \le i \le n$, $j \in N$ for J_n) and the multiplication table

$$a_{ij} \cdot a_{kl} = \frac{1}{2}(a_{il} + a_{kj}). \tag{3}$$

In standard IR-algebras it is convenient to use the following multiplication diagram:



For calculations in standard IR-algebras it is convenient to use the following correlations (see [1]):

$$(x \cdot y) \cdot z = \frac{1}{2}(x \cdot z + y \cdot z),$$

$$(x \cdot y) \cdot (z \cdot t) = \frac{1}{4}(x \cdot z + x \cdot t + y \cdot z + y \cdot t), \tag{4}$$

which are implemented for the basis elements J.

And in fact,

$$(a_{i_1i_2}\cdot a_{j_1j_2})\cdot a_{k_1k_2}=\frac{1}{2}(a_{i_1j_2}+a_{j_1i_2})\cdot a_{k_1k_2}=\frac{1}{4}(a_{i_1k_2}+a_{k_1j_2}+a_{j_1k_2}+a_{k_1i_2})=\frac{1}{2}(a_{i_1i_2}\cdot a_{k_1k_2}+$$
$$+a_{j_1j_2}\cdot a_{k_1k_2});$$

$$(a_{i_1i_2}\cdot a_{j_1j_2})\cdot(a_{k_1k_2}\cdot a_{l_1l_2})=\frac{1}{4}(a_{i_1j_2}+a_{j_1i_2})(a_{k_1l_2}+a_{l_1k_2})=\frac{1}{8}(a_{i_1l_2}+a_{i_1k_2}+a_{k_1j_2}+a_{l_1j_2}+a_{j_1l_2}+$$
$$+a_{j_1k_2}+a_{k_1i_2}+a_{l_1i_2})=\frac{1}{4}(a_{i_1i_2}\cdot a_{k_1k_2}+a_{i_1i_2}\cdot a_{l_1l_2}+a_{j_1j_2}\cdot a_{k_1k_2}+a_{j_1j_2}\cdot a_{l_1l_2}).$$

Let's note that the correlations (4) are not valid for arbitrary elements. For example,

$$((x-y)\cdot z)\cdot t=(x\cdot z-y\cdot z)\cdot t\underset{(4)}{=}\frac{1}{2}(x\cdot t-y\cdot t)\neq\frac{1}{2}(x-y)\cdot t+\frac{1}{2}z\cdot t,$$

for $x=a_{11}$, $y=a_{22}$, $z=t=a_{33}$.

Using the correlation (4) it is easy to check that in standard IR-algebras the identity (1) is valid.

1.2. General results

Let's review the main results of this work.

The multiplication table (3) shows that the standard IR-algebras are algebras with genetic realization (see [5]). Among the algebras with genetic realization the class of Bernstein algebras holds an important position.

Let's remind that the Bernstein algebra over the field F is a commutative algebra J with a non-zero algebra homomorphism $\omega:J\to F$, satisfying the identity

$$x^2\cdot x^2=\omega(x)^2x^2.$$

These algebras were introduced by P. Holgate [9]. It is known [5] that the algebra J can be represented as

$$J=Fe\oplus N,$$

where $N=Ker\omega$ and e is idempotent, $n^2\cdot n^2=0$ for all $n\in N$. If $ch(F)\neq 2$, then

$$N=U\oplus Z,$$

where $U=\{u\in N\mid e\cdot u=\frac{1}{2}u\}$, $Z=\{z\in N\mid e\cdot z=0\}$. On the algebra N the following Bernstein graduation is defined:

$$U^2\subseteq Z,\ Z^2\subseteq U,\ U\cdot Z\subseteq U.$$

A Bernstein algebra is called Jordan, if it also satisfies the Jordan identity

$$(x^2 \cdot y) \cdot x = x^2 \cdot (y \cdot x).$$

The Jordan Bernstein algebras were first introduced by P. Holgate [10], who proved that the genetic algebras for the simple Mendel inheritance should are special Jordan algebras. Later this result was generalized by A. Wörs-Busekros [11]. It was shown that finite-dimensional Bernstein algebras with zero multiplication in N are special Jordan algebras. Also in the paper [11] the necessary and sufficient conditions for a Bernstein algebra to be Jordan were obtained: $z^2 = 0$ and $N = U \oplus Z$ is nil-index 3 algebra.

Jordan Bernstein algebras play an important role in the theory of Bernstein algebras (see [5], [12], [13]).

Definition 1. *Bernstein algebra $B = Fe \oplus U \oplus Z$ is called <u>annihilator</u> algebra if Z coincides with annihilator of the algebra B, i.e. $Z = Ann(B)$.* It is easy to note that in the annihilator algebras $Z^2 = N^3 = 0$. That is why all annihilator algebras are Jordan Bernstein algebras.

In the Section 2 of the present work we prove that all standard IR-algebras are Bernstein algebras (Theorem 1) and furthermore, the class of standard IR-algebras coincides with the class of annihilator Bernstein algebras of a special type (Theorem 2).

We will denote by $F(c_i;\ i \in I)$ the free F-module generated by the elements $c_i,\ i \in I$. The Section 2 describes the annihilator of a standard IR-algebra J_∞. It is found that

$$Ann(J_\infty) = F((a_{11} - a_{ij})^2;\ i > 1, j > 1)$$

and the following isomorphism of the modules takes place:

$$J_\infty / Ann(J_\infty) \simeq F(a_{ij}; i = 1\, or\ j = 1), \text{ where } Ann(J_\infty / Ann(J_\infty)) = 0 \text{ (Lemma 1).}$$

Let's adjoin a formal unit 1 to the algebra $J_\infty / Ann(J_\infty)$. It is shown that the algebra

$$J^\# = F \cdot 1 + J_\infty / Ann(J_\infty)$$

is a Jordan algebra of symmetrical bilinear form over F (Lemma 3).

The Section 3 of this work proves the speciality of the variety of all IR-algebras (Theorem 3).

Let us denote by $B_\infty = F \cdot 1 + V$ the Jordan algebra of non-degenerate symmetric bilinear form $f : V \times V \to F$, where V is an infinite vector space over F. Using the defining identities of the variety $Var(B_\infty)$ (see the results by S. Vasilovsky [7]), it is proved that the variety IR is a proper subvariety $Var(B_\infty)$, i.e.

$$IR \subsetneq Var(B_\infty).$$

In view of the results [8], the variety $Var(B_\infty)$ is special. That is why any commutative algebra satisfying the identity (1) is a special Jordan algebra. Let's note that the class of Jordan Bernstein algebras is not special (see [6]).

In the Section 4 we will investigate the free IR-algebras.

Definition 2. *Associative algebra A is called <u>splicing algebra</u> or S-algebra if it meets the identity*

$$x[y,z]t = 0. \tag{5}$$

It is easy to note that all standard splicing algebras are S-algebras. Let S denote the variety of all S-algebras. Let $S[x]$ be a free algebra in the variety S with set of free generators $X = \{x_1, ..., x_n, ...\}$.

The Theorem 4 finds the basis of the identities of a standard splicing algebra C_∞. It is found that

$$S = Var(C_\infty),$$

i.e. all the identities of a standard splicing algebra C_∞ follow from the identity (5).

Let $IR[X]$ denote a free algebra in the variety IR with generating set X.

It is shown that $S[X]$ is an associative envelope algebra for $IR[X]$. In the lemmas 5, 6, 7 the basis and multiplication tables are built for the free algebras $S[X]$ and $IR[X]$.

The Theorem 5 proves that all the identities of a standard IR-algebra J follow from the identity (1), i.e.

$$IR = Var(J_\infty).$$

In the Section 5 we describe the annihilator of a free algebra $IR[X]$ and prove that

$$Ann(IR[X]/Ann(IR[X])) = 0.$$

The Theorem 6 proves that the following isomorphism of F-modules takes place:

$$IR[X] \simeq F[X] \oplus D,$$

where $F[X]$ is a free associative-commutative algebra with generating set $X = \{x_1, ..., x_n, ...\}$, $D = D(IR[X])$ - an associator ideal of the algebra $IR[X]$. Furthermore, $D = D_0 \oplus D_1$, where $D_1 = Ann(IR[X])$, $D_0^{\ 2} \subseteq D_1$ and $D^3 = 0$. From this we can conclude that $D = M(IR[X])$, where $M(IR[X])$ - is the McCrimmon radical of $IR[X]$.

In the Section 6 we will investigate the basis of the identities of standard IR-algebras J_n and $J_{n,m}$. It is proved that all the variety of IR-algebras is generated by a minimal nontrivial standard algebra $J_{1,2}$. This algebra has the basis $a = a_{11},\ b = a_{12}$ and the multiplication table $a^2 = a,\ b^2 = b,\ a \cdot b = \frac{1}{2}(a+b)$. Consequently,

$$Var(J_{1,2}) = Var(J_{n,m}) = Var(J_{\infty}) = IR\ ,\ (n,m) \neq (1,1)\,.$$

To prove the speciality of the variety IR in the Section 3, we used two complicated results: the description of the identities of variety $Var(B_{\infty})$ [7] and the speciality of the variety $Var(B_{\infty})$ [8]. The basis and the multiplication table for free algebras $S[X]$ and $IR[X]$ built in the Section 4 allow us to prove the specialty of the variety IR by means of a rather simple method.

It is found that the variety IR possesses the following property:

$$IR[X] = HS[X]\,,$$

i.e. the algebra $IR[X]$ coincides with a Jordan algebra of symmetric elements of associative envelope $S[X]$ under the standard involution. We will call the varieties of Jordan algebras satisfying this property the _reflective varieties_.

In the Section 7 we will prove the following theorem – any reflective variety of Jordan algebras is special (Theorem 8).

2. Annihilator Bernstein algebras

In this Section we will prove that the class of standard IR-algebras coincides with the class of annihilator Bernstein's algebras of a special type.

2.1. Annulets of the standard IR-algebras

Let $b_{ij} = a_{11} + a_{ij} - 2a_{11} \cdot a_{ij} = (a_{11} - a_{ij})^2 = a_{11} + a_{ij} - a_{1j} - a_{i1}$ for all $i, j > 1$. At the multiplication diagram the elements b_{ij} form cells $i \times j$:

$$\begin{array}{c|ccc|c} a_{11} & \cdot & \cdot & \cdot & a_{1j} \\ \hline \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \hline a_{i1} & \cdot & \cdot & \cdot & a_{ij} \end{array}$$

Let us first prove that $b_{ij} \in Ann(J_{\infty})$ for all $i, j > 1$. Indeed, for any basis element $a = a_{kl}$ of the algebra J_{∞}, we have

$$b_{ij} \cdot a = (a_{11} + a_{ij} - 2a_{11} \cdot a_{ij}) \cdot a \underset{(4)}{=} a_{11} \cdot a + a_{ij} \cdot a - a_{11} \cdot a - a_{ij} \cdot a = 0$$

Further, for any $a = \sum_{k,l} \alpha_{kl} a_{kl}$ from J_∞, we'll have

$$b_{ij} \cdot \sum_{k,l} \alpha_{kl} a_{kl} = \sum_{k,l} \alpha_{kl} (b_{ij} a_{kl}) = 0 .$$

It is easy to see that the set $\{b_{ij};\ i > 1,\ j > 1\}$ is linearly independent over F. By definition of b_{ij}, we have

$$\sum_{i,j>1} \alpha_{ij} b_{ij} = 0 \Rightarrow \sum_{i,j>1} \alpha_{ij} (a_{11} + a_{ij} - a_{1j} - a_{i1}) = 0 \Rightarrow \alpha_{ij} = 0 \text{ for any } i > 1,\ j > 1 .$$

We denote by $L(c_i; i \in I)$ the F-module generated by the elements $C_i, i \in I$, i.e.

$$L(c_i; i \in I) = \{\sum_i \alpha_i c_i \mid \alpha_i \in F\} .$$

We will call $L(c_i; i \in I)$ a linear envelope of the set $\{c_i; i \in I\}$.

In view of the proven, we have $L(b_{ij};\ i > 1, j > 1) = F(b_{ij};\ i > 1, j > 1)$. Let $I = F(b_{ij};\ i > 1, j > 1)$.

Lemma 1. *The following isomorphism of the F-modules takes place:*

$$J_\infty / Ann(J_\infty) \simeq F(a_{ij};\ where\ i = 1\ or\ j = 1),$$

where $Ann(J_\infty) = I$ *and* $Ann(J_\infty / Ann(J_\infty)) = 0$.

Proof. We first prove the isomorphism $J_\infty / I \simeq F(a_{ij};\ where\ i = 1\ or\ j = 1)$.

Let's consider an arbitrary basis element $a_{ij};\ i > 1,\ j > 1$. In view of the definition of the elements $b_{ij};\ i > 1,\ j > 1$, we have $a_{ij} = b_{ij} - a_{11} + a_{1j} + a_{i1}$ hence $a_{ij} \in L(a_{11}, a_{1j}, a_{i1}) + I$. Consequently,

$$J_\infty / I = L(a_{ij};\ where\ i = 1\ or\ j = 1) .$$

We now prove that the elements $a_{11}, a_{1i}, a_{j1},\ where\ i > 1\ and\ j > 1$ are linearly independent in J_∞ / I. Let's suggest the contrary, then in the algebra J_∞ we'll have $0 \neq a = \alpha_{11} a_{11} + \sum_{i=2}^{n} \beta_{1i} a_{1i} + \sum_{i=2}^{n} \gamma_{i1} a_{i1} \in I \subseteq Ann(J)$.

Then $a \cdot a_{11} = \alpha_{11} a_{11} + \frac{1}{2}(\sum_{i=2}^{n} \beta_{1i} (a_{11} + a_{1i})) + \frac{1}{2}(\sum_{i=2}^{n} \gamma_{i1} (a_{11} + a_{i1})) = 0$.

Consequently, $\beta_{1i}=0$ and $\gamma_{i1}=0$ for all $i\geq 2$. Then $a=\alpha_{11}a_{11}$, but a_{11} is an idempotent, so $\alpha_{11}=0$. The obtained contradiction proves that

$$J_\infty / I \simeq F(a_{ij};\ where\ i=1\ or\ j=1).$$

Let now $a\in Ann(J)$. Then $a=\alpha_{11}a_{11}+\sum_{i=2}^{n}\beta_{1i}a_{1i}+\sum_{i=2}^{n}\gamma_{i1}a_{i1}+v$, where $v\in I$. As $a\in Ann(J_\infty),\ v\in I\subseteq Ann(J_\infty)$, then $a\cdot a_{11}=0$. But in this case $\alpha_{11}=0,\ \beta_{1i}=\gamma_{i1}=0,\ i\geq 2$. So, $a=v\in I$ and $Ann(J_\infty)=I$.

Let now $a\in Ann(J_\infty / I)$, then $a\cdot a_{11}=0$. Similar arguments apply to this case, we'll get $a=0$, i.e. $Ann(J_\infty / I)=0$. This proves the lemma.

Let's now build a structure of Bernstein Jordan algebra on the algebra J_∞. To do this, we'll introduce the following notations:

$e=a_{11};$
$e_{1i}=a_{11}-a_{1i},\ i>1;$
$e_{i1}=a_{11}-a_{i1},\ i>1;$
$U=F(e_{1i};e_{i1};i>1);$
$Z=Ann(J_\infty)=F(b_{ij};i>1,\ j>1).$

Lemma 2. *In the algebra J_∞ the following relations are valid:*

1. $J_\infty = F_e\oplus U\oplus Z$ *- the direct sum of the F-modules;*

2. $e^2=e,\ e\cdot u=\frac{1}{2}u,\ u\in U$ *and* $e\cdot z=0,\ z\in Z$*;*

3. $U^2\subseteq Z$*;*

4. $N^3=0$*, where* $N=U\oplus Z$.

Proof. (1.) follows from the Lemma 1.

(2.) Calculating with the multiplication table (3) we obtain:

$e^2={a_{11}}^2=a_{11}=e,$
$e\cdot z=0,$
$e\cdot e_{1i}=(a_{11}-a_{1i})\cdot a_{11}=a_{11}-\frac{1}{2}a_{1i}-\frac{1}{2}a_{11}=\frac{1}{2}(a_{11}-a_{1i})=\frac{1}{2}e_{1i},$
$e\cdot e_{i1}=(a_{11}-a_{i1})\cdot a_{11}=a_{11}-\frac{1}{2}a_{i1}-\frac{1}{2}a_{11}=\frac{1}{2}(a_{11}-a_{i1})=\frac{1}{2}e_{i1}.$

Let now $u=\sum_{i>1}\alpha_{1i}e_{1i}+\sum_{i>1}\beta_{i1}e_{i1}$, then $e\cdot u=\sum_{i>1}\alpha_{1i}(e_{1i}\cdot e)+\sum_{i>1}\beta_{i1}(e_{i1}\cdot e)=\frac{1}{2}u$.

(3.) According to the distributivity of the multiplication it is sufficient to check the statement 3 on the basis elements U.

We have

$e_{1i}\cdot e_{1j}=(a_{11}-a_{1i})\cdot(a_{11}-a_{1j})=a_{11}-\frac{1}{2}a_{1j}-\frac{1}{2}a_{11}-\frac{1}{2}a_{11}-\frac{1}{2}a_{1i}+\frac{1}{2}a_{1j}+\frac{1}{2}a_{1i}=0$ for all $i>1,\ j>1$. Analogously, $e_{i1}\cdot e_{j1}=0;\ i,j>1$.

Let's now prove that $e_{i1}\cdot e_{1j}\in Ann(J_\infty)$ for all $i,j>1$.

Indeed,

$e_{i1}\cdot e_{1j}=(a_{11}-a_{i1})\cdot(a_{11}-a_{1j})=a_{11}-\frac{1}{2}a_{1j}-\frac{1}{2}a_{11}-\frac{1}{2}a_{i1}-\frac{1}{2}a_{11}+\frac{1}{2}a_{ij}+\frac{1}{2}a_{11}=\frac{1}{2}(a_{11}+$

$+a_{ij}-a_{1j}-a_{i1})=\frac{1}{2}(a_{11}+a_{ij}-2a_{11}\cdot a_{ij})=\frac{1}{2}b_{ij}\in Ann(J).$

Hence, $U^2\subseteq Z$.

(4.) We have the following sequence of containments:

$N^3=(U\oplus Z)^2\cdot N\subseteq U^2\cdot N\subseteq Z\cdot N=0$.

The lemma is proved.

Let's adjoin a formal unit 1 to the algebra $J_\infty/Ann(J_\infty)$. Let $J^\#$ denote the obtained algebra as

$$J^\#=F\cdot 1+J_\infty/Ann(J_\infty).$$

Lemma 3. *The algebra $J^\#$ is a Jordan algebra of symmetric bilinear form over F.*

Proof. Let $e_{11}=(2e-1)$. From the Lemma 1 we conclude that

$$J^\#=F\cdot 1+F(l_{11},l_{1i},l_{j1};\ i,j>1).$$

It follows from the proof of the Lemma 2 that

$e_{11}{}^2=(2e-1)^2=4e-4e+1=1;$
$e_{1i}\cdot e_{1j}=e_{i1}\cdot e_{j1}=e_{i1}\cdot e_{1j}=0,\ i,j>1.$

Consequently, $\{e_{11},e_{1i},e_{j1};\ i,j>1\}$ is a standard basis of a Jordan algebra of symmetric bilinear form over F. This proves the Lemma.

Observe that the bilinear form defined in the Lemma 3 is degenerate.

Theorem 1. *Standard IR-algebras* $J_\infty,\ J_{n,m},\ J_n$ *are Bernstein Jordan algebras with the following* (U,Z) *-graduation:*

1. $J_\infty = F_e \oplus U \oplus Z$, *where* $U = F(e_{1i}, e_{j1};\ i,j>1)$, $Z = Ann(J) = F(b_{ij};\ i,j>1)$, $e = a_{11}$;

2. $J_{n,m} = F_e \oplus U_{n,m} \oplus Z_{n,m}$, *where* $U_{n,m} = F(e_{1j}, e_{i1};\ 1<j\le m,\ 1<i\le n)$, $Z_{n,m} = Ann(U_{n,m}) = F(b_{ij};\ 1<i\le n,\ 1<j\le m)$, $e = a_{11}$;

3. $J_n = F_e \oplus U_n \oplus Z_n$, *where* $U_n = F(e_{1j}, e_{i1};\ 1<i\le n,\ j>1)$, $Z_n = F(b_{ij};\ 1<i\le n,\ j>1)$, $e = a_{11}$;

and nontrivial homomorphism $\omega : J_\infty(J_{n,m}, J_n) \to F$, *which is defined by:*

$$\forall_x = \alpha e + u + z \in J,\ \text{where}\ \alpha \in F,\ u \in U(U_{n,m},\ U_n),\ z \in Z(Z_{n,m},\ \mathrm{Z}_n);$$

$$\omega(x) = \alpha .$$

Proof. It follows from the Lemma 2 that the introduced graduations are Bernstein graduations (see [6]) and $Z^2 = 0,\ N^3 = 0$. In view of the proposition 3.1 [6], the algebras $J_\infty,\ J_{n,m},\ J_n$ are Bernstein Jordan algebras. The theorem is proved.

2.2. Annihilator algebras of the type (V, W)

Let's note that $Z = Ann(J)$ for all Bernstein algebras $J = Fe + U + Z$ defined in the Theorem 1. In accordance with the definition 1 in the Section 1.2, such algebras are called annihilator algebras.

Our next goal is to build a free algebra in the class of annihilator algebras.

Let $I = (X)_J$ denote the ideal of the algebra J generated by $X \subseteq J$.

Let $BJ = BJ[X;Y]$ - a free (U,Z)-graded Bernstein algebra (see [6]) from U-generators $X = \{x_1,...,x_n,...\}$ and Z-generators $Y = \{y_1,...,y_n,...\}$. It is easy to see that the algebra $J = F\cdot e + BJ[X;Y]/I$, where $I = (w\cdot u \mid w \in Z,\ u \in BJ[X;Y])_{BJ}$ is a free annihilator algebra. We will denote by $AnnBJ[X;Y]$ the nucleus of the Bernstein algebra J, i.e. $AnnBJ[X;Y] = BJ[X;Y]/I$.

Let now $X = VUW$, where $V = \{v_1,...,v_n,...\}$, $W = \{w_1,...,w_n,...\}$ - some sets and $Y = \varnothing$.

Definition 3. *The annihilator algebra* $J = Fe + AnnBJ[X;\varnothing]/I$, *where* $I = (v_i \cdot v_j, w_i \cdot w_j;\ i,j \in N)_{AnnBJ[X;\varnothing]}$ *is called the annihilator algebra of the type (V,W).*

Proposition 1. *The annihilator algebra of the type* (V,W) *has the basis* $\{e, v_i, w_j, v_i w_j;\ i,j \in N\}$ *and the following multiplication table:*

$$\begin{aligned} & e^2 = e,\ e\cdot u_i = \frac{1}{2}u_i,\ e\cdot w_j = \frac{1}{2}w_j, \\ & v_i \cdot v_j = w_i \cdot w_j = 0,\ v_i \cdot w_j = v_i w_j, \\ & v_i w_j \in Z = Ann(J), \textit{ for all } i,j \in N. \end{aligned} \tag{6}$$

Proof. In view of the properties of free (U,Z)-graded Bernstein algebra (see [6]), we have $Y = \varnothing \Rightarrow U^2 = Z$. But $Z = Ann(J)$, since $U = F(v_i, w_j; i,j \in N)$ and $Z = F(u_i w_j;\ i,j \in N)$. The proposition is proved.

Theorem 2. *The class of standard IR-algebras coincides with the class of annihilator algebras of the type* (V,W). *An annihilator algebra of the type* (V,W) *is isomorphic*

to J_∞ *if* $|V| = |W| = \infty$,

to J_n *if* $|V| = n,\ |W| = \infty$,

to $J_{n,m}$ *if* $|V| = n,\ |W| = m$.

Proof. We first consider a standard IR-algebra J_∞. It follows from the Theorem 1, that $J = Fe \oplus U \oplus Z$, where $U = F(e_{1i}, e_{j1};\ i,j > 1)$, $Z = Ann(J) = F(b_{ij};\ i,j > 1)$, $e = a_{11}$. Let us introduce the following notations:

$$v_i = e_{1i},\ w_i = e_{i1},\ i > 1,$$
$$V = \{v_i;\ i > 1\},\ W = \{w_i;\ i > 1\}.$$

Let's prove that the algebra J_∞ is an annihilator algebra of the type (V,W).

From the proof of the Lemma 2 it follows that

$$v_i \cdot v_j = w_i \cdot w_j = 0, \textit{ for } i,j > 1;$$

$$v_i \cdot w_j = e_{1i} \cdot e_{j1} = \frac{1}{2} b_{ij} \in Z.$$

We conclude from the Lemma 1 that the variety $\{v_i \cdot w_j = \frac{1}{2} b_{ij}, i,j > 1\}$ is linearly independent over F. Hence, the algebra J_∞ has the basis and the multiplication table (6). Consequently, it is isomorphic to an annihilator algebra of the type (V,W), where $|V| = |W| = \infty$.

Conversely let us consider an annihilator algebra B of the type (V,W) with a basis and a multiplication table (6).

Let

$a_{11} = e;$

$a_{1i} = e - v_{i-1},\ i \geq 2;$

$a_{j1} = e - w_{j-1},\ j \geq 2;$

$a_{ij} = 2v_{i-1}w_{j-1} - v_{i-1} - w_{j-1} + e,\ i,j \geq 2.$

At first it is necessary to note that $B = F(a_{ij};\ i,j \in N)$, i.e. $a_{ij};\ i,j \in N$ is the basis of the algebra B. Now we will found a multiplication table in this basis:

$a_{11}{}^2 = a_{11};$

$a_{11} \cdot a_{1i} = e - \frac{1}{2}v_{i-1} = \frac{1}{2}a_{11} + \frac{1}{2}a_{1i};$

$a_{11} \cdot a_{j1} = e - \frac{1}{2}w_{j-1} = \frac{1}{2}a_{11} + \frac{1}{2}a_{j1};$

$a_{11} \cdot a_{ij} = -\frac{1}{2}v_{i-1} - \frac{1}{2}w_{j-1} + e = \frac{1}{2}a_{1j} + \frac{1}{2}a_{i1},\ i,j \geq 2;$

$a_{1i} \cdot a_{1j} = (e - v_{i-1}) \cdot (e - v_{j-1}) = e - \frac{1}{2}v_{i-1} - \frac{1}{2}v_{j-1} = \frac{1}{2}a_{1i} + \frac{1}{2}a_{1j},\ i,j \geq 2;$

Analogously,

$$a_{ij} \cdot a_{kl} = (-v_{i-1} - w_{j-1} + e) \cdot (-v_{k-1} - w_{l-1} + e) = v_{i-1}w_{l-1} + v_{k-1}w_{j-1} - \frac{1}{2}v_{i-1} - \frac{1}{2}w_{j-1} - \frac{1}{2}v_{k-1} -$$
$$-\frac{1}{2}w_{l-1} + e = \frac{1}{2}(a_{il} + a_{kj}),\ i,j,k,l \geq 2.$$

Therefore, the algebra B has the multiplication table (3) in the basis $a_{ij};\ i,j \in N$. Hence, it is isomorphic to a standard IR-algebra J_∞. The same proof works for the cases $|V| = n,\ |W| = m$ and $|V| = n,\ |W| = \infty$. This proves the theorem.

3. The specialty of the variety IR

In this Section we will prove that the variety IR is a proper subvariety of the variety of special algebras $Var(B_\infty)$ [8].

Lemma 4. *In the variety IR the following identities are valid:*

$zU_{x_1x}R_y = zR_xU_{x_1y};$ (7)

$2xD_{x,z} \cdot y = yD_{x^2,z};$ (8)

$xD_{y,z}R_t + xD_{z,t}R_y + xD_{t,y}R_z = 0$; (9)

$3xD_{y,z} \cdot t = tD_{x\cdot y,z} + tD_{y,x\cdot z}$; (10)

$xD_{y,z}U_{t_1 t} = 0$; (11)

$((x\cdot z)U_{y,t} + (y\cdot t)U_{x,z} - (x\cdot t)U_{y,z} - (y\cdot z)U_{x,t})\cdot u = 0$; (12)

$zU_{x^2,y} = zR_x U_{x,y}$; (13)

$[x,y]^2 \cdot z = 0,$ *where* $[x,y]^2 = 2yD_{y,x^2} - 4yD_{y,x}\cdot x$; (14)

$zR_{x_1}...R_{x_n}U_{y,y} = zR_{x_{6(1)}}...R_{x_{6(n)}}U_{y,y}$, *for any* $\sigma \in Sn$. (15)

Furthermore, the identities (1), (7), (8) are equivalent in the variety of commutative algebras.

Proof.

$(1) \Rightarrow (7)$.

$$2(zU_{x_1 x}R_y - zR_x U_{x\cdot y}) = 2(2z\cdot x\cdot x\cdot y - z\cdot x^2\cdot y - z\cdot x\cdot x\cdot y - z\cdot x\cdot y\cdot x + (z\cdot x)\cdot(x\cdot y)) \underset{(J)}{=}$$
$$= 4z\cdot x\cdot x\cdot y - 2z\cdot x^2\cdot y + 2(z\cdot x)\cdot(x\cdot y) - 2z\cdot x\cdot x\cdot y + x^2\cdot y\cdot z - x^2\cdot(y\cdot z) -$$
$$-2(x\cdot z)\cdot(x\cdot y) = 2z\cdot x\cdot x\cdot y - 2z\cdot x^2\cdot y + x^2\cdot y\cdot z - x^2\cdot(y\cdot z) \underset{(1)}{=} 0,$$

where J is a Jordan identity $2z\cdot x\cdot y\cdot x + x^2\cdot y\cdot z = 2(z\cdot x)\cdot(x\cdot y) + x^2\cdot(y\cdot z)$.

$(7) \Rightarrow (1)$. In view of the above proved it is enough to check that $(7) \Rightarrow (J)$. Substituting $y = x$ into (7), we have $2z\cdot x\cdot x\cdot x - (z\cdot x^2)\cdot x = 2z\cdot x\cdot x\cdot x - (z\cdot x)\cdot x^2$, i.e. $(z\cdot x^2)\cdot x = (z\cdot x)\cdot x^2$. Therefore, the identities (1) and (7) are equivalent for commutative algebras.

$(1) \Leftrightarrow (8)$. It is easy to notice that the identity (8) is a D-operator representation of the identity (1):

$$x^2\cdot y\cdot z - x^2\cdot(z\cdot y) + 2x\cdot z\cdot x\cdot y - 2x^2\cdot z\cdot y = 2xD_{z,x}\cdot y + yD_{x^2,z} = 0.$$

$(1) \Leftrightarrow (9)$. Rewriting the left part of (9) in the terms of U-operator, we have:

$xU_{y,z} - yU_{x,z} = x\cdot y\cdot z + x\cdot z\cdot y - (y\cdot z)\cdot x - (x\cdot y)\cdot z - (y\cdot z)\cdot x + (x\cdot z)\cdot y = 2zD_{x,y}$.
Consequently,

$$2zD_{x,y} = xU_{y,z} - yU_{x,z}. \qquad (16)$$

Hence,

$4xD_{y,z}R_t \underset{(16)}{=} 2(yU_{z,x} - zU_{y,x})R_t \underset{(7)}{=} ((y\cdot z)U_{x,t} + (x\cdot y)U_{z,t} - (z\cdot y)U_{x,t} - (z\cdot x)U_{y,t}) =$
$= (x\cdot y)U_{z,t} - (x\cdot z)U_{y,t}.$

Writing $\sigma f(y,z,t) = f(y,z,t) + f(z,t,y) + f(t,y,z)$ gives

$4\sigma(xD_{y,z}R_t) = \sigma(x\cdot y)U_{z,t} - \sigma(x\cdot z)U_{y,t} = \sigma(x\cdot y)U_{z,t} - \sigma(x\cdot y)U_{t,z} = 0$.

$(1) \Rightarrow (10)$. We have

$3xD_{y,z}\cdot t = (2xD_{y,z} - yD_{z,x} - zD_{x,y})\cdot t = (xD_{y,z} + yD_{x,z} + xD_{y,z} + zD_{y,x})\cdot t \underset{(8)}{=} tD_{x\cdot y,z} + tD_{y,z\cdot x}$

.

$(1) \Rightarrow (11)$. It follows from the proved identities (8), (10), that

$3xD_{y,z}\cdot t^2 \underset{(10)}{=} t^2D_{x\cdot y,z} + t^2D_{y,x\cdot z} = -(x\cdot y)D_{z,t^2} - zD_{t^2,x\cdot y} - yD_{x\cdot z,t^2} - (x\cdot z)D_{t^2,y} \underset{(8)}{=} -2tD_{z,t}R_{(x\cdot y)} -$
$-2tD_{t,x\cdot y}R_z - 2tD_{x\cdot z,t}R_y - 2tD_{t,y}R_{(x\cdot z)} \underset{(9)}{=} 2(tD_{x\cdot y,z} + tD_{y,x\cdot z})R_t \underset{(10)}{=} 6xD_{y,z}\cdot t\cdot t.$

This gives

$$xD_{y,z}U_{t,t} = 0.$$

$(1) \Rightarrow (12)$. From (7), (11) we obtain

$2((x\cdot z)U_{y,t} + (y\cdot t)U_{x,z} - (x\cdot t)U_{y,z} - (y\cdot z)U_{x,t})R_u \underset{(7)}{=} ((x\cdot z\cdot y)U_{t,u} + (x\cdot z\cdot t)U_{y,u} +$
$+(y\cdot t\cdot x)U_{z,u} + (y\cdot t\cdot z)U_{x,u} - (x\cdot t\cdot y)U_{z,u} - (x\cdot t\cdot z)U_{y,u} - (y\cdot z\cdot x)U_{t,u} - (y\cdot z\cdot t)U_{x,u}) =$
$= (zD_{x,y}U_{t,u} + xD_{z,t}U_{y,u} + tD_{y,x}U_{z,y} + yD_{t,z}U_{x,u}) \underset{(11)}{=} 0.$

$(1) \Rightarrow (13)$. Since IR is a Jordan variety, the known Jordan identity is valid in IR (e.g., see [4]):

$$zU_{x,x}R_y + zU_{x^2,y} = 2zR_xU_{x,y}.$$

From this,

$$zU_{x^2,y} = 2zR_xU_{x,y} - zU_{x,x}R_y \underset{(7)}{=} zR_xU_{x,y}.$$

$(1) \Rightarrow (14)$. We have

$[x,y]^2 = 2yD_{y,x^2} - 4yD_{y,x}\cdot x \underset{(16)}{=} yU_{x^2,y} - x^2U_{y,y} - 2yU_{x,y}R_x + 2xU_{y,y}R_x \underset{(13),(7)}{=} (x\cdot y)U_{x,y} -$
$-x^2U_{y,y} - (x\cdot y)U_{x,y} - y^2U_{x,x} + 2(x\cdot y)U_{x,y} = 2(x\cdot y)U_{x,y} - x^2U_{y,y} - y^2U_{x,x}.$

Then, $[x,y]^2\cdot u = (2(x\cdot y)U_{x,y} - x^2U_{y,y} - y^2U_{x,x})\cdot u \underset{(12),\, z=y,\, t=x}{=} 0$.

$(1) \Rightarrow (15)$. It suffices to show that:

$$zD_{x_1,x_2}R_{y_1}...R_{y_n}U_{y,y} = 0,$$

where the operators $R_{y_1}...R_{y_n}$ can be missing.

We induct on n. The base of induction is the identity (11), when $R_{y_1}...R_{y_n}$ are missing. Now $3zD_{x_1,x_2}R_{y_1}...R_{y_n}U_{y,y} \underset{(10)}{=} y_1 D_{z\cdot x_1,x_2}R_{y_2}...R_{y_n}U_{y,y} + y_1 U_{x_1,z\cdot x_2}R_{y_2}...R_{y_n}U_{y,y} = 0$ by induction. This proves the lemma.

Theorem 3. *The variety IR is a proper subvariety $Var(B_\infty)$, i.e.*

$$IR \subsetneq Var(B_\infty).$$

The variety IR is special, i.e. any commutative algebra satisfying the identity (1) is a special Jordan algebra.

Proof. By S. Vasilovsky's results [7], the variety $Var(B_\infty)$ is defined by the following identities:

$$zD_{[x,y]^2,t} = 0,$$
$$\sigma(x^2 D_{y,z}R_t - 2xD_{y,z}R_t R_x) = 0.$$

It is easily seen that the first identity is the consequence of the identity (14), and the second is the consequence of the identity (9).

Therefore,

$$IR \subseteq Var(B_\infty).$$

It is obvious that (14) is not valid in B_∞. Hence, $IR \neq Var(B_\infty)$. By the Theorem 3.1 [8], $Var(B_\infty)$ is a special variety. Therefore the variety IR is special. This proves the theorem.

4. Free S and IR-algebras

We will denote by $IR[X]$ the free algebra in the variety IR of generating set $X = \{x_1,...,x_n,...\}$. In this Section we will build a basis and a multiplication table of the free algebra $IR[X]$ and will prove that the variety of all IR-algebras is generated by a standard IR-algebra J_∞, i.e.

$$IR = Var(J_\infty).$$

4.1. The variety of the algebras of splicing.

Let $Ass[X]$, $SJ[X]$ be free associative, free special Jordan algebras. Let's remind (Definition 2) that the associative algebra A is called S-algebra if it meets the identity (5)

$$x[y,z]t = 0 .$$

It is easy to check that the standard algebras of splicing are S-algebras. By (2), $a_{i_1 j_1} a_{i_2 j_2} a_{i_3 j_3} a_{i_4 j_4} \underset{(2)}{=} a_{i_1 j_4}$, hence, $a_{i_1 j_1} [a_{i_2 j_2}, a_{i_3 j_3}] a_{i_4 j_4} = 0$.

Let us denote by S the variety of all S-algebras, and let $S[X]$ be a free algebra in this variety.

Let's build a basis and multiplication table for a free algebra S. We will define an *ordering operator* $< >: Ass[X] \to SJ[X]$ by a rule:

If $u = x_{s_1} ... x_{s_m}$ is a monomial from $Ass[X]$ then

$$< u > = x_{i_1} \cdot ... \cdot x_{i_m} \in SJ[X],$$

where $i_1 \leq ... \leq i_m$ and the set $\{s_1, ..., s_m\}$ and $\{i_1, ..., i_m\}$ coincide with respect to repetitors of all the symbols. Then we will extend the ordering operator on the algebra $Ass[X]$ by linearity: if $f = \sum \alpha_i u_i$, where $\alpha_i \in F$, u_i are monomials, then $< f > = \sum \alpha_i < u_i >$. For example, $< x_1 x_2 x_1 + 3 x_3 x_1 > = x_1 \cdot x_1 \cdot x_2 + 3 x_1 \cdot x_3$.

By definition, the operation of multiplication of the elements of the algebra $Ass[X]$ and consequently of the algebra $SJ[X]$ within the brackets $< >$ is associative-commutative. Therefore, for any $v_1, ..., v_n \in Ass[X]$ and $\sigma \in S_n$, we have

$$< v_1 \cdot ... \cdot v_n > = < v_1 v_2 ... v_n > = < v_{\sigma(1)} ... v_{\sigma(n)} > .$$

Let's consider the algebra $A[X]$ with the basis $B = \{x_i, x_i x_j = x_i < 1 > x_j,\ x_i < u > x_j\}$, where $x_i \in X$ and $< u >$ runs over all ordered monomials of $SJ[X]$, 1 – a formal unit; with the multiplication table:

$$x_i \rhd x_j = x_i < 1 > x_j;$$

$$x_i \rhd x_j < u > x_k = x_i < u > x_j \rhd x_k = x_i < u x_j > x_k;$$

$$x_i < u > x_j \rhd x_k < v > x_l = x_i < u v x_k x_j > x_l,\ where\ i, j, k, l \in N.$$

Lemma 5. *The algebras* $S[X]$ *and* $A[X]$ *are isomorphic.*

Proof. It follows from the definition of the multiplication and ordering operator in the algebra $A[X]$, that the algebra $A[X]$ is an S-algebra. Consequently, the identical mapping $\tau : X \to X$ has a unique extension to canonical homomorphism $\tau : S[X] \to A[X]$. From (5) we conclude that

$$S[X]=L[B].$$

It is clear that the images of the basis elements from B under homomorphism τ are linearly independent in $A[X]$. Hence,

$$S[X]=F[B].$$

Consequently, the algebras $S[X]$ and $A[X]$ are isomorphic. This proves the lemma.

Theorem 4. *$S=Var(C_\infty)$, i.e. all the identities of the standard algebra of splicing C_∞ follow from the identity (5).*

Proof. The algebra C_∞ is an S-algebra, hence $Var(C_\infty)\subseteq S$. Let's prove the contrary inclusion.

Let a homogenous polynomial $f=\sum_{i,j}\alpha_{ij}x_i<u_{ij}>x_j\in S[X]$, where $\alpha_{ij}\in F$, be an identity on C_∞. Consider the mapping $\varphi:x_i\to a_{ii}$ and extend it up to the homomorphism $\varphi:S[X]\to C_\infty$. Such extension exists due to the fact that $S[X]$ is a free algebra in the variety S. Then,

$$\varphi(f)=\sum_{i,j}\alpha_{ij}a_{ii}<\varphi(u_{ij})>a_{jj}=\sum_{i,j}\alpha_{ij}a_{ij}=0.$$

Hence, $\alpha_{ij}=0$ for all i,j. Consequently, $f=0$ in the algebra $S[X]$ and f is the consequence of the identity (5). This proves the theorem.

4.2. Basis and multiplication table of the algebra IR[X]

We will denote by $D=D(IR[X])$ the associator ideal of the algebra $IR[X]$, i.e. the ideal generated by all Jordan associators aD_{bc}, where $a,b,c\in IR[X]$.

Proposition 2. *In the algebra $IR[X]$ the following relations are valid:*

$$D=L(aD_{b,c};\, where\; a,b,c\in IR[X]); \tag{17}$$

$$uU_{x,y}=<u>U_{x,y}; \tag{18}$$

$$<u>U_{x,y}R_z=\frac{1}{2}<ux>U_{y,z}+\frac{1}{2}<uy>U_{x,z}; \tag{19}$$

$$<u>U_{x,y}\cdot<v>U_{z,t}=\frac{1}{4}(<uvyz>U_{x,t}+<uvyt>U_{x,z}+<uvxt>U_{y,z}+<uvxz>U_{y,t}), \tag{20}$$

where $u,v,x,y,z,t\in IR[X]$ and u,v can be formal units.

Proof. The relation (17) follows immediately from the identity (10).

It is evident that $u = <u> + d$, where $d \in D$. Hence, $uU_{x,y} = <u>U_{x,y} + dU_{x,y} \underset{(17),(11)}{=} <u>U_{x,y}$.

We have

$$<u>U_{x,y}R_z \underset{(7)}{=} \frac{1}{2}<u>R_xU_{y,z} + \frac{1}{2}<u>R_yU_{x,z} \underset{(18)}{=} \frac{1}{2}<ux>U_{y,z} + \frac{1}{2}<uy>U_{x,z}.$$

Analogously,

$$4<u>U_{x,y} \cdot <v>U_{z,t} \underset{(7),(18)}{=} 2<ux>U_{y,<v>}U_{z,t} + 2<uy>U_{x,<v>}U_{z,t} = <uxvz>U_{y,t} +$$
$$+ <uxt>U_{y,<v>\cdot z} + <uxvt>U_{y,z} + <uxz>U_{y,<v>\cdot t} - <uxv>U_{y,z\cdot t} - <uxzt>U_{y,v} +$$
$$+ <yuvz>U_{x,t} + <uyt>U_{x,<v>\cdot z} + <uyvt>U_{v,z} + <uyz>U_{x,<t>\cdot t} - <uyv>U_{x,z\cdot t} -$$
$$- <uyzt>U_{x,v} \underset{(13),(18)}{=} <uvyz>U_{x,t} + <uvyt>U_{x,z} + <uvxt>U_{y,z} + <uvxz>U_{y,t}.$$

This proves the proposition.

Let's consider a subset $B \subseteq IR[X]$ of the following type:

$$B = \{x_i,\ x_i \cdot x_j = <1>U_{x_i,x_j},\ <u>U_{x_i,x_j}\},$$

where $x_i, x_j \in X$, $<u>$ runs over all ordered monomials of $SJ[X]$, 1 – a formal unit. In view of the relations (19), (20) the linear envelope $L(B)$ is a sub-algebra $IR[X]$, i.e. it is closed under multiplication.

Lemma 6. *The set* B *is the basis of the algebra* $IR[X]$.

Proof. We first show that $IR[X] = L(B)$. Let's consider an arbitrary homogenous polynomial $w \in IR[X]$. We will prove by induction on $\deg(w)$ that $w \in L(B)$. If $\deg(w) \le 3$, then $w \in L(B)$ by definition of B. Let's assume that $\deg(w) = n$, $n \ge 4$ and all homogenous monomials of the length $< n$ belong to $L(B)$.

It is well known that the algebra of multiplication $R(SJ[X])$ is generated by the set of operators $\{R_{x_i},\ U_{x_i,x_j};\ where\, x_i, x_j \in X\}$. Hence, $w \in L(uU_{x_i,x_j}, vR_{x_i})$, where $x_i, x_j \in X$, u, v are homogenous monomials of $IR[X]$ of degree $n-2$ and $n-1$ correspondingly.

It follows from the relation (18) that $uU_{x_i,x_j} = <u>U_{x_i,x_j} \in L(B)$. By induction assumption,

$$vR_{x_i} = (\sum_{j,k} \alpha_{j,k} <v_{j,k}>U_{x_j,x_k})R_{x_i} = \sum_{j,k} \alpha_{jk} <v_{jk}>U_{x_i,x_k}R_{x_i},$$

where $\alpha_{jk} \in F$.

From (19) we have

$$< v_{j,k} > U_{x_j,x_k} \cdot R_{x_i} \underset{(19)}{=} \frac{1}{2} < v_{jk}x_j > U_{x_k,x_i} + \frac{1}{2} < v_{jk}x_k > U_{x_j,x_i} \in L(B).$$

Consequently, $w \in L(B)$ and $IR[X] = L(B)$.

Let's now prove that the set B is linearly independent over F, i.e. that

$$IR[X] = F(B).$$

Let's suppose that $f = \sum_{i,j} \alpha_{ij} < u_{ij} > U_{x_i,x_j} = 0$ in the algebra $IR[X]$, where f is a homogenous polynomial and $\alpha_{ij} = \alpha_{ji} \in F$.

Consider the mapping $\varphi : x_i \to a_{ii}$ and extend it to homomorphism $\varphi : IR[X] \to J_\infty$. Such extension exists due to the fact that J_∞ is a IR-algebra. Then in the algebra J_∞ we have:

$$\varphi(f) = \sum_{i,j} \alpha_{ij} \varphi(< u_{ij} >) U_{a_{ii},a_{jj}} \underset{(4)}{=} \frac{1}{2} \sum_{i,j} \alpha_{ij} (\varphi(< u_{ij} >) \cdot a_{jj} + a_{ii} \cdot a_{jj} + \varphi(< u_{ij} >) \cdot a_{ii} + a_{ii} \cdot a_{jj} -$$

$$-\varphi(< u_{ij} >) \cdot a_{ii} - \varphi(< u_{ij} >) \cdot a_{jj}) = \sum_{i,j} \alpha_{ij} (a_{ii} \cdot a_{jj}) = \frac{1}{2} \sum_{i,j} \alpha_{ij} (a_{ij} + a_{ji}) = 0.$$

Consequently, $\alpha_{ij} = 0$ for all i, j and $f = 0$. So, it follows that $IR[X] = F(B)$. The lemma is proved.

Observe that the multiplication table on the basis B of the algebra $IR[X]$ is defined by the relations (19), (20) in case when $x, y, z, t \in X$, which follows from the Lemma 6.

4.3. Associative envelope for $IR[X]$ and basis of the identities of the algebra $J\infty$

On the algebra $S[X]$ it is defined a standard involution *, which is set on the basis words in accordance with the following rule:

$$x_i^* = x_i,\ (x_i x_j)^* = x_j x_i,\ (x_i < u > x_j)^* = x_j < u > x_i,$$

where $x_i, x_j \in X$ and u runs over all ordered monomials of $SJ[X]$ and linearly extends to the whole algebra $S[X]$.

Let $HS[X]$ be a Jordan algebra $H(S[X], *)$ of symmetrical elements of the algebra $S[X]$ relatively to $*$. And let $JS[X]$ be a Jordan subalgebra $S[X]^{(+)}$ generated by the set X.

Proposition 3. $HS[X] = JS[X]$.

Proof. It is obvious that $JS[X] \subseteq HS[X]$. Let's prove the contrary inclusion. Consider an arbitrary homogenous element $f \in HS[X]$. Then $f = \sum_{i,j} \alpha_{ij} x_i < u_{ij} > x_j$ and $f^* = f$, where $\alpha_{ij} \in F$. It follows that $\sum_{i,j} \alpha_{ij} x_i < u_{ij} > x_j = \sum_{i,j} \alpha_{ij} x_j < u_{ij} > x_i$. Consequently, $\alpha_{ij} = \alpha_{ji}$ for all i, j.

Finally,
$f = \frac{1}{2}(f + f^*) = \frac{1}{2}\sum_{i,j} \alpha_{ij}(x_i < u_{ij} > x_j + x_j < u_{ij} > x_i) = \sum_{i,j} \alpha_{ij} < u_{ij} > U_{x_i,x_j} \in JS[X]$.
This proves the proposition.

Lemma 7. *The algebra* $S[X]$ *is an associative envelope for the algebra* $IR[X]$.

Proof. It suffices to prove that the algebras $IR[X]$ and $JS[X]$ are isomorphic. Let us consider the set

$$B = \{x_i, x_i \cdot x_j =< 1 > U_{x_i,x_j},\ < u > U_{x_i,x_j}\},$$

where $x_i, x_j \in X$ and u runs over all ordered monomials of $SJ[X]$ and constitutes the basis of the algebra $JS[X]$. It follows from the Lemma 6 and the proof of the Proposition 3 that B is a basis of $JS[X]$.

Let us find the multiplication table in this basis:

$x_i \circ x_j =< 1 > U_{x_i,x_j}$;

$x_i \circ < u > U_{x_k,x_l} = \frac{1}{2} x_i \circ (x_k < u > x_l + x_l < u > x_k) = \frac{1}{4}(x_i < x_k u > x_l + x_k < u x_l > x_i +$

$+ x_i < x_l u > x_k + x_l < u x_k > x_i) = \frac{1}{2} < u x_k > U_{x_i,x_l} + \frac{1}{2} < u x_l > U_{x_k,x_l}$;

analogously,

$< u > U_{x_i,x_j} \circ < v > U_{x_l,x_k} = \frac{1}{4}(< uvx_i x_k > U_{x_j,x_l} + < uvx_i \cdot x_l > U_{x_j,x_k} + < uvx_j x_k > U_{x_i,x_l} +$

$+ < uvx_j x_l > U_{x_i,x_k})$.

Thus, the basis and the multiplication tables of the algebras $IR[X]$ and $JS[X]$ coincide, what proves the lemma.

Theorem 5. $IR = Var(J_\infty)$, *i.e. all the identities of a standard IR-algebra* J_∞ *follow from the identity (1).*

Proof. We have $Var(J_\infty) \subseteq IR$. Let's prove a contrary inclusion. Let the homogenous polynomial $f = \sum_{i,j} \alpha_{ij} < u_{ij} > U_{x_i,x_j}$, where $\alpha_{ji} = \alpha_{ij} \in F$, be an identity on the algebra J_∞.

Let's consider the homomorphism $\varphi : IR[X] \to J_\infty$, defined in the Lemma 6. Then

$$\varphi(f) = \frac{1}{2}\sum_{i,j} \alpha_{ij}(a_{ij} + a_{ji}) = 0 .$$

Hence, $\alpha_{ij} = 0$ for all i, j and $f = 0$ in the algebra $IR[X]$. Consequently, the defining identities of the algebras $IR[X]$ and J_∞ coincide. This proves the theorem.

5. Annihilator of the free algebra IR[X]

In this section we will describe the generators of the $Ann(IR[X])$. The examples of non-zero elements of $Ann(IR[X])$ were found in the Lemma 4:

$$(x \cdot z)U_{y,t} + (y \cdot t)U_{x,z} - (x \cdot t)U_{y,z} - (y \cdot z)U_{x,t} \underset{(12)}{\in} Ann(IR[X]);$$

$$yD_{y,x^2} - 2yD_{y,x} \cdot x \underset{(13)}{\in} Ann(IR[X]),$$

where $x = x_1, y = x_2, z = x_3, t = x_4$.

Let $n = n(x, y, z, t, u) =< uzt > U_{x,y} + < uxy > U_{z,t} - < uzy > U_{x,t} - < uxt > U_{z,y}$, where $x, y, z, t, u \in IR[X]$ and u can be a formal unit.

Let's check that $n \in Ann(IR[X])$. Indeed,

$$2n \cdot v =< uzt > U_{x,y}R_v + < uxy > U_{z,t}R_v - < uzy > U_{x,t}R_v - < uxt > U_{z,y}R_v \underset{(19)}{=} < uztx > U_{y,v} +$$
$$+ < uzty > U_{x,v} + < uxyz > U_{t,v} + < uxyt > U_{z,v} - < uzyx > U_{t,v} - < uzyt > U_{x,v} - < uxtz > U_{y,v} -$$
$$- < uxty > U_{z,v} = 0.$$

Lemma 8. *In the algebra $IR[X]$ the following relations are valid:*

1. $Ann(IR[X]) = L(n(x, y, z, t, u))$, where $x, y, z, t \in X$ and u runs over all ordered monomials of $SJ[X]$ including a formal 1.

2. $Ann(IR[X] / Ann(IR[X])) = 0$.

Proof. 1. From what has already been proved, $L = L(n(x, y, z, t, u)) \subseteq Ann(IR[X])$. Let's now prove the contrary inclusion. Consider an arbitrary non-zero homogenous element $f \in Ann(IR[X])$. It is obvious that $\deg(f) \geq 4$. Let's decompose f by the basis of $IR[X]$:

$$f = f(x_1,...,x_n) = \sum_{i,j=1}^{n} \alpha_{ij} < u_{ij} > U_{x_i,x_j} ,$$

where $\alpha_{ij} = \alpha_{ji} \in F$, $\deg(u_{ij}) \geq 2$. Renumerating, if required, the generators $x_1,...,x_n$ in $f = f(x_1,...,x_n)$, we will have

$$\deg_{x_1}(f) \geq \deg_{x_2}(f) \geq ... \geq \deg_{x_n}(f) \geq 1.$$

If $\deg_{x_1}(f) \geq 2$, then due to the definition of the elements n we have

$$< ux_1^2 > U_{x_i,x_j} + < ux_i x_j > U_{x_1,x_1} - < ux_1 x_i > U_{x_j,x_1} - < ux_1 x_j > U_{x_i,x_1} \in L .$$

Consequently, $f = \sum_{i=1}^{n} \alpha_{1i} < u_i > U_{x_1,x_i} + u$, where $\alpha_{1i} \in F$, $u \in L$.

Then

$$2f \cdot x_1 \underset{(7)}{=} 2\alpha_{11} < u_1 x_1 > U_{x_1,x_1} + \sum_{i=2}^{n} \alpha_{1i} (< u_i x_1 > U_{x_1,x_i} + < u_i x_i > U_{x_1,x_1}) = 0 .$$

It follows from the view of the basis words of the algebra $IR[X]$ that $\alpha_{1i} = 0$ for $2 \leq i \leq n$, and hence $\alpha_{11} = 0$. Consequently, $f = u \in L$.

If $\deg_{x_1}(f) = 1$ then $f(x_1,...,x_n)$ is a multilinear polynomial.

We have

$$< ux_1 x_2 > U_{x_i,x_j} + < ux_i x_j > U_{x_1,x_2} - < ux_i x_2 > U_{x_1,x_j} - < ux_1 x_j > U_{x_2,x_j} \in L ,$$

$$< ux_1 x_3 > U_{x_2,x_4} + < ux_2 x_4 > U_{x_1,x_3} - < ux_2 x_3 > U_{x_1,x_4} - < ux_1 x_4 > U_{x_2,x_3} \in L .$$

Therefore,

$$f = \sum_{i=2}^{n} \alpha_i < u_{1i} > U_{x_1,x_i} + \beta < v > U_{x_2,x_3} + u ,$$

where $\alpha_i, \beta \in F$, $u \in L$.

Then

$$\begin{aligned} 2f \cdot x_{n+1} &= \sum_{i=2}^{n} \alpha_i (< u_{1i} x_1 > U_{x_i,x_{n+1}} + < u_{1i} x_i > U_{x_1,x_{n+1}}) + \beta < vx_2 > U_{x_3,x_{n+1}} + \\ &+ \beta < vx_3 > U_{x_2,x_{n+1}} = 0 \end{aligned} .$$

It follows from the view of the basic words of $IR[X]$ that $\alpha_i = 0$ at $2 \le i \le n$, hence, $\beta = 0$, too. Consequently, $f = u \in L$.

2. Let us consider an arbitrary non-zero homogenous element $f = f(x_1,...,x_n) \in IR[X]/Ann(IR[X])$. From what has already been proved, it follows that $\deg(f) \ge 4$. Let $f \in Ann(IR[X]/Ann(IR[X]))$. Then we have $f \cdot u \cdot v = 0$ for any $u, v \in IR[X]$. Proceeding analogously as above, we come to two cases:

either $f = \sum_{i=1}^{n} \alpha_{1i} < u_i > U_{x_1,x_i}$, where $\alpha_{1i} = \in F$. But in this case $f \cdot x_1 \cdot x_1 = 0$ and $\alpha_{1i} = 0$ for $2 \le i \le n$, and hence $\alpha_{11} = 0$;

or $f = \sum_{i=2}^{n} \alpha_i < u_{1i} > U_{x_1,x_i} + \beta < v > U_{x_2,x_3}$, where $\alpha_i, \beta \in F$. But in this case $f \cdot x_{n+1} \cdot x_{n+2} = 0$ and $\alpha_i = 0$ for $2 \le i \le n$, and hence $\beta = 0$, too, what makes a contradiction. Consequently, $Ann(IR[X]/Ann(IR[X])) = 0$. The Lemma is proved.

Corollary. $Ann(IR[X]) \subseteq D = D(IR[X])$

Proof. It follows from the Lemma 8 that it is sufficient to prove that $\forall x, y, z, t, u \in IR[X]$ $\; n = n(x, y, z, t, u) \in D$.

We have

$$n =< uzt > U_{x,y} - < uzy > U_{x,t} + < uxy > U_{z,t} - < uxt > U_{z,y} \underset{(18)}{=} < uz > (R_t U_{x,y} - R_y U_{x,t}) +$$
$$+ < ux > (R_y U_{z,t} - R_t U_{z,y}).$$

But $4xD_{y,z}R_t \underset{(16)}{=} 2(yU_{z,x} - zU_{y,x})R_t \underset{(7)}{=} x(R_y U_{z,t} - R_z U_{y...})$, thus, $n \in D$. This proves the corollary.

Choose a basis $< u >$ in the free associative commutative algebra $F[X]$ of generation $X = \{x_1,..., x_n,...\}$, where $< u >$ runs over all ordered monomials of $SJ[X]$. Then $F[X] = F(< u >)$.

Define $D_1 = Ann(IR[X])$. By the corollary, $D_1 \subseteq D$. Let D_0 be a direct complement of F-module D_1 in D, then $D = D_0 \oplus D_1$.

Theorem 6. *The following isomorphism of F-modules takes place: $IR[X] \simeq F[X] \oplus D$, where $D = D_0 \oplus D_1$, $D_0^2 \subseteq D_1$, $D_1 = Ann(IR[X])$ and $D^3 = 0$.*

Proof. It is obvious that $u =< u > +d$, where a monomial $u \in F[X]$ and $d \in D$. Therefore, $IR[X] \simeq F[X] + D$. If $< u >\in F[X] \cap D$, then it is clear that $<< u >>=< u >= 0$. Hence, $IR[X] \simeq F[X] \oplus D$.

Let's verify that $D_0^2 \subseteq D_1$. By (17) it is sufficient to prove that

$\forall a,b,c,x,y,z \in IR[X] \quad aD_{b,c} \cdot xD_{y,z} \in D_1$.

We have

$$3aD_{b,c} \cdot xD_{y,z} = -3aD_{b,c}D_{y,z} \cdot x + 3(aD_{b,c} \cdot x)D_{y,z} \underset{(10)}{=} -3aD_{b,c}D_{y,z} \cdot x + xD_{a\cdot b,c}D_{y,z} + +xD_{b,c\cdot a}D_{y,z}.$$

On the other hand,

$$\forall a,b,c,y,z \in IR[X] \qquad 2aD_{b,c}D_{y,z} \underset{(16)}{=} yU_{(aD_{b,c}),z} - zU_{(aD_{b,c}),y} \underset{(13)}{=} \frac{1}{2}(< yab > U_{c,z} + + < yc > U_{a\cdot b,z} - < yac > U_{b,z} - < yb > U_{a\cdot c,z} - < zab > U_{c,y} - < zc > U_{a\cdot b,y} + < zac > U_{b,y} + + < zb > U_{a\cdot c,y}) \underset{(13)}{=} \frac{1}{2}(< ayb > U_{c,z} + < azc > U_{y,b} - < ayc > U_{b,z} - < azc > U_{b,y}) + +\frac{1}{4}(< yca > U_{b,z} - < yba > U_{c,z} - < zca > U_{b,y} + < zba > U_{c,y}) \underset{(20)}{\in} Ann(IR[X])$$

Hence, $aD_{b,c} \cdot xD_{y,z} \in D_1$.

Further, $D^3 = (D_0 \oplus D_1)^3 \subseteq D_0^3 \subseteq D_1 \cdot D_0 = 0$.

The theorem is proved.

Corollary. $D = M(IR[X])$, *where* $M(IR[X])$ *is a MacCrimmon radical of* $IR[X]$.

Proof. By the Theorem 6, $\forall d \in D$, $z \in IR[X]$, $zU_{d,d} \in Ann(IR[X]) \subseteq \mathbb{Z}(IR[X])$, where $\mathbb{Z}(IR[X])$ is an ideal of $IR[X]$ generated by all absolute zero divisors. That is why $D \subseteq M(IR[X])$. But the algebra $IR[X]/D \simeq F[X]$ is a nongenerated algebra. Consequently, $D = M(IR[X])$. The corollary is proved.

6. Basis of the identities of the algebras J_n and $J_{n,m}$.

Let's find the basis of the algebra $J_{1,2}$. This algebra has the basis $a = a_{11}$, $b = a_{12}$ and the following multiplication table:

$$a^2 = a, \quad b^2 = b, \quad a \cdot b = \frac{1}{2}(a+b).$$

Lemma 9. $Var(J_\infty) = Var(J_{1,2})$.

Proof. Since $J_{1,2}$ is a subalgebra J_∞, then $Var(J_{1,2}) \subseteq Var(J_\infty)$. Let's suppose that $Var(J_{1,2}) \neq Var(J_\infty)$. Then there exists a nonzero homogenous multilinear polynomial $f = f(x_1,...,x_n) \in IR[X]$, which is an identity on $J_{1,2}$. It is evident that $\deg f \geq 3$. Let's decompose it over the basis of the algebra $IR[X]$:

$$f = \sum_{1 \leq i < j \leq n} \alpha_{ij} < u_{ij} > U_{x_i, x_j} ,$$

where $\alpha_{ij} \in F$. Reindexing if necessary the generators of the alternation in f, we can assume that $\alpha_{1,2} \neq 0$. Let $\overline{f} = f|_{x_1 = v_1, ..., x_n = v_n}$ is the value of polynomial f in the algebra $J_{1,2}$ for $x_1 = v_1, ..., x_n = v_n$, where $v_i \in J_{1,2}$.

Let $x_1 = a_{12}, x_2 = ... = x_n = a_{11}$. Then

$$\overline{f} = \sum_{i=2}^{n} \alpha_{1i} < \overline{u_{1i}} > U_{a_{12}, a_{11}} + \sum_{2 \leq i \leq j \leq n} \alpha_{ij} < \overline{u_{ij}} > U_{a_{11}, a_{11}} \underset{(4)}{=} \sum_{i=2}^{n} \alpha_{1i} (a_{12} \cdot a_{11}) + \sum_{2 \leq i \leq j \leq n} \alpha_{ij} a_{11} =$$

$$= \frac{1}{2} \sum_{i=2}^{n} \alpha_{1i} (a_{12} + a_{11}) + \sum_{2 \leq i \leq j \leq n} \alpha_{ij} a_{11} = 0$$

Consequently,

$$\sum_{i=2}^{n} \alpha_{1i} = 0 .$$

Similarly,

$$\sum_{j=1, j \neq i}^{n} \alpha_{ij} = 0 .$$

Let $x_1 = a_{11} + 2a_{12}$, $x_2 = a_1 + 3a_{12}, x_3 = ... = x_n = a_{11}$. Then

$$\overline{f} = \alpha_{12} < \overline{u_{12}} > U_{(a_{11} + 2a_{12}), (a_1 + 3a_{12})} + \sum_{i=3}^{n} \alpha_{1i} < \overline{u_{1i}} > U_{(a_{11} + 2a_{12}), a_{11}} + \sum_{i=3}^{n} \alpha_{2i} < \overline{u_{2i}} > U_{(a_{11} + 3a_{12}), a_{11}} +$$

$$+ \sum_{3 \leq i \leq j \leq n} \alpha_{ij} < \overline{u_{ij}} > U_{a_{11}, a_{11}} = 0$$

By (4):

$$< \overline{u_{12}} > U_{(a_{11} + 2a_{12}), (a_{11} + 3a_{12})} = (a_{11} + 2a_{12}) \cdot (a_{11} + 3a_{12}) = a_{11} + \frac{3}{2} (a_{12} + a_{11}) + (a_{11} + a_{12}) + 6a_{12} =$$

$$= \frac{7}{2} a_{11} + \frac{17}{2} a_{12};$$

$$< \overline{u_{1i}} > U_{(a_{11}+2a_{12}),a_{11}} = (a_{11}+2a_{12})\cdot a_{11} = 2a_{11}+a_{12};$$

$$< \overline{u_{2i}} > U_{(a_{11}+3a_{12}),a_{11}} = (a_{11}+3a_{12})\cdot a_{11} = a_{11}+\frac{3}{2}(a_{11}+a_{12}) = \frac{5}{2}a_{11}+\frac{3}{2}a_{12};$$

$$< \overline{u_{ij}} > U_{a_{11},a_{11}} = a_{11}\cdot a_{11} = a_{11}.$$

Consequently,

$$\alpha_{12}(\frac{7}{2}a_{11}+\frac{17}{2}a_{12})+\sum_{i=3}^{n}\alpha_{1i}(2a_{11}+a_{12})+\sum_{i=3}^{n}\alpha_{2i}(\frac{5}{2}a_{11}+\frac{3}{2}a_{12})+\sum_{3\le i\le j\le n}\alpha_{ij}a_{11}=0.$$

Hence,

$$\alpha_{12}(7a_{11}+17a_{12})+\sum_{i=3}^{n}\alpha_{1i}(4a_{11}+2a_{12})+\sum_{i=3}^{n}\alpha_{2i}(5a_{11}+3a_{12})+\sum_{3\le i\le j\le n}2\alpha_{ij}a_{11}=0.$$

Since $\sum_{i=2}^{n}\alpha_{1i}=0$, we have $\alpha_{12}(3a_{11}+15a_{12})+\sum_{i=3}^{n}\alpha_{2i}(5a_{11}+3a_{12})+\sum_{3\le i\le j\le n}2\alpha_{ij}a_{11}=0$.

Since $\sum_{\substack{1\le i\le n\\ i\ne 2}}\alpha_{ij}=0$, then $12\alpha_{12}a_{12}+\sum_{i=3}^{n}\alpha_{2i}(2a_{11})+\sum_{3\le i\le j\le n}2\alpha_{ij}a_{11}=0$. Consequently, $\alpha_{12}=0$. This contradicts our assumption. This proves the lemma.

Theorem 7. *The variety $Var(J_{1,1})=K$ is the variety of associative commutative algebras and has the determining identity $xD_{y,z}=0$;*

The variety $Var(J_{1,2})=Var(J_{n,m})=Var(J_{\infty})=IR$ and has the determining identity (1).

Proof. It is obvious that $Var(J_{1,1})=K$. From the chain of inclusion $J_{1,2}\subseteq J_{n,m}\subseteq J_{\infty}$, we get $Var(J_{1,2})\subseteq Var(J_{n,m})\subseteq Var(J_{\infty})$.

From the Lemma 9 and the Theorem 5 it follows that $Var(J_{1,2})=Var(J_{n,m})=Var(J_{\infty})=IR$. The theorem is proved.

7. Reflexive varieties of Jordan algebras

The proof of the specialty of variety IR in the Section 3 required application of two rather complex results: description of identities of the variety $Var(B_{\infty})$ [7] and the specialty of the variety $Var(B_{\infty})$ [8]. The construction basis and the multiplication table of the algebras $S[X]$ and $IR[X]$ (Section 4) allow us to easily prove the specialty of the variety IR.

Let $\mathfrak{M}$ be some homogenous variety of Jordan algebras. Let a free algebra $M[X]$ in $\mathfrak{M}$ be a special Jordan algebra and $A[X]$ be some associative envelope algebra for $M[X]$.

There exists a natural involution $*$ on $A[X]$, which acts on the monomials by $(x_1...x_n)^* = x_n...x_1$, and linearly extends over the whole algebra $A[X]$.

We will denote by $HA[X]$ a Jordan algebra of symmetric elements of $A[X]$ in regard to $*$. It is obvious that $M[X] \subseteq HA[X]$.

Definition. *The variety M is called reflexive if $M[X] = HA[X]$ for some associative envelope algebra $A[X]$ of $M[X]$.*

Theorem 8. *Any reflexive variety of Jordan algebras is special.*

Proof. It suffices to prove that all homomorphic images of $M[X]$ are special Jordan algebras. Let's consider the algebra $J \simeq M[X]/I$ and let's prove that J is a special algebra. In accordance with the Cohn lemma [8] it is sufficient to show that $\hat{I} = (I)_{A[X]} \cap M[X] = I$, where $(I)_{A[X]}$ is the ideal of the algebra $A[X]$ generated by the set I.

Let $f \in \hat{I}$. Then $f = \sum_i \alpha_i a_i u_i b_i$, where a_i, b_i are the monomials of $A[X]$ or formal units, and $u_i \in I$. Since $f \in M[X]$, then $f^* = f$. Consequently, in view of reflexivity of the M, we have $f = \frac{1}{2}\sum_i \alpha_i (a_i u_i b_i + b^* u_i a_i^*) = \sum_i h_i(u_i, X)$, where $h_i(u_i, X)$ are Jordan polynomials of u_i and X. Therefore, $f \in I$ and $\hat{I} = I$. This proves the theorem.

Corollary. *The variety IR is reflexive and, consequently, it is special.*

Proof. The reflexivity of IR is proved in the Proposition 3, and the corollary is proved.

It would be interesting to describe the reflexive varieties of Jordan algebras.

Acknowledgements

I thank professor I. Hentzel for bringing my interest in investigation of use of Jordan algebras in mathematical genetics and professor M. Bremner for setting an interesting question on specialty of Jordan algebras arising from intermolecular recombination.

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