MODELS
OF THE LIE ALGEBRA F_4

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Abstract

We describe the models of the exceptional Lie algebra F_4 which are based on its semisimple subalgebras of rank 4. The underlying fact is that the reductive subalgebras of maximal rank of a simple Lie algebra induce a grading on this algebra by means of an abelian group, in such a way that the nontrivial components of the grading are irreducible modules.

1 Introduction

Although the exceptional Lie algebras were discovered more than one hundred years ago and a lot of mathematicians have been interested in them, these algebras are so surprising that it can be always expected a new way of looking at them.

The most known model of F_4, the 52-dimensional simple Lie algebra, is due to Chevalley and Schafer [ChS], who showed that F_4 is the set of derivations of the Albert algebra, the only exceptional simple Jordan algebra. This fact led Tits, among other authors, to study the relationship between Jordan algebras and the remaining exceptional simple Lie algebras, which were constructed in a unified way [T]. This construction is reflected in a magic square, in particular, it allows us to look at F_4 through its subalgebra $G_2 \oplus A_1$.

Since then, several authors have given different versions of the magic square, like Vinberg [V], Barton and Sudbery [BaSu], Landsberg and Manivel [LMa3] and Elduque [E]. The last three ones are related to the triality, and give a point of view of F_4 based on its subalgebra D_4, acting on its three irreducible 8-dimensional representations, closely related to the mentioned construction in [ChS] (D_4 is the subalgebra of the derivations annihilating the three idempotents in the Albert algebra J). Notice that this construction of F_4 is a grading in the Klein group: the identity component is D_4, and the other three components are the natural module and the two half-spin ones.

Jacobson's proof that $\operatorname{Der} J$ is a central simple Lie algebra of dimension 52 [Ja4, ch IX, sect 11] suggests a different model of F_4. If $J = J_0 \oplus J_{\frac{1}{2}} \oplus J_1$ is the Peirce decomposition relative to a primitive idempotent e, then $\mathfrak{h} = \{d \in \operatorname{Der} J \mid d(e) = 0\}$ is isomorphic to the Lie algebra of linear transformations in the nine-dimensional vector space B (being B the subspace of J_0 of

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the elements of generic trace 0, so that $J_0 = F(1-e) \oplus B$) which are skew relative to q, the quadratic form given by $v \cdot v = q(v)(1-e)$ if $v \in B$. Thus $\mathfrak{h}$ is a Lie algebra of type B_4, and its set of restrictions to $J_{\frac{1}{2}}$ is an irreducible set of linear transformations such that $\mathrm{Der}\, J \cong \mathfrak{h} \oplus J_{\frac{1}{2}}$. That is, we have a symmetric decomposition, equivalently a $\mathbb{Z}_2$-grading, with even part B_4 and odd part an irreducible $\mathfrak{h}$-module. The related construction (for instance, [B, p. 190]) describes the bracket in $F_4 \approx \mathrm{so}(9) \oplus S_9$, for S_9 the spinor representation of the orthogonal algebra so(9), by means of the bracket in so(9), the spin action $\mathrm{so}(9) \times S_9 \to S_9$ and the map $S_9 \times S_9 \to \mathrm{so}(9)$ obtained by dualizing the spin action.

The other Lie triple system with standard embedding F_4 provides another $\mathbb{Z}_2$-grading, based on $A_1 \oplus C_3$. In fact this construction is a particular case of the following result which involves all the exceptional Lie algebras (to see [E2, th. 4.4], or [YA] for characteristic cero): Let U be a two-dimensional vector space, φ a nonzero skew-symmetric bilinear form on U, V a vector space endowed with a skew-symmetric bilinear form $<\ ,\ >: V \times V \to F$ and $d: V \times V \to \mathrm{End}_F(V)$, $(u, v) \mapsto d_{u,v}$ a symmetric map such that

$$
\begin{gathered}
< d_{u,v}(x), y > + < x, d_{u,v}(y) >= 0 \\
[d_{u,v}, d_{x,y}] = d_{d_{u,v}(x),y} + d_{x,d_{u,v}(y)} \\
d_{x,y}(z) - d_{y,z}(x) = 2 < z, x > y - < y, z > x - < x, y > z
\end{gathered}
$$

for any $x, y, z, u, v \in V$ (that is, V is a symplectic triple system, as in [YA]). Let us take the $\mathbb{Z}_2$-graded algebra

$$
\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \quad \begin{cases} \mathfrak{g}_0 = \mathrm{sp}(U, \varphi) \oplus \mathrm{span}\langle d_{u,v} \mid u, v \in V \rangle \\ \mathfrak{g}_1 = U \otimes_F V \end{cases} \tag{1}
$$

where $\mathfrak{g}_0$ is a subalgebra, $[\mathfrak{g}_0, \mathfrak{g}_1]$ is given by the natural action as a module and the product of odd elements is $[a \otimes u, b \otimes v] =< u, v > \varphi_{a,b} + \varphi(a, b) d_{u,v}$ for $\varphi_{a,b} = \varphi(-, a)b + \varphi(-, b)a$. Then $\mathfrak{g}$ is a $\mathbb{Z}_2$-graded Lie algebra. Concretely, the symmetric pairs $(\mathfrak{g}, \mathfrak{g}_0) = (G_2, 2A_1)$, $(F_4, A_1 \oplus C_3)$, $(E_6, A_1 \oplus A_5)$, $(E_7, A_1 \oplus D_6)$ and $(E_8, A_1 \oplus E_7)$ are particular cases taking $V = \left\{ \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \mid a, b \in J \right\}$, where J is the Jordan algebra F, $H_3(F)$, $H_3(F + F)$, $H_3(\mathbb{Q})$ and $H_3(\mathbb{O})$ respectively. Here, if $(X, -)$ is an algebra with involution, $H_3(X) := \{x \in \mathrm{Mat}_{3\times 3}(X) \mid \bar{x} = x^t\}$.

The last three models of F_4 above have a common fact: they are gradings of F_4 such that the identity component is a semisimple subalgebra of rank 4. One of our objectives is to show that this is not a coincidence, but a general situation. For a reductive subalgebra $\mathfrak{h}$ of a simple Lie algebra $\mathcal{L}$, the completely reducible $\mathfrak{h}$-module $\mathcal{L}$ is decomposed as a sum of $\mathfrak{h}$-irreducible modules. In general, it is a fruitful idea to build an algebra from simpler constituents, but in Section 3 we will prove that if besides $\mathfrak{h}$ is of maximal rank, such decomposition as a sum of irreducible modules provides a grading of $\mathcal{L}$ over an abelian group, verifying interesting properties which enable to obtain models of $\mathcal{L}$ in terms of $\mathfrak{h}$ and its representations: the nonzero homogeneous components of the grading are irreducible submodules and the homomorphisms between the tensor products of the components are generated by the projections of the bracket in $\mathcal{L}$. Hence the bracket in $\mathcal{L}$ is determined by these homomorphisms up to some scalars, which can be obtained imposing the Jacobi identity to be satisfied. Some models related to $\mathbb{Z}$- and $\mathbb{Z}_m$-gradings appear in [OV, Ch 5, §2] making use of a similar philosophy.

The main aim of this paper is to describe the models of F_4 related to the remaining semisimple subalgebras of rank 4, since the preceding arguments show that it is possible to do it using only linear algebra.

The maximal rank reductive subalgebras are regular, so that they have been studied in the complex field by Dynkin [Dy, Ch II, §5]. In the case in which they are too maximal subalgebras, they are obtained from the affine Dynkin diagrams ([Ka, p. 54-55] and [F]) and it is known that they are in correspondence to the finite order automorphisms, studied by Kac [Ka, Ch 8]; providing in this way a grading of the algebra over a finite cyclic group. We will review these basic facts in the sections 2 and 3, because our models are obtained applying them several times.

After computing in Section 2 the seven semisimple maximal rank subalgebras of F_4, we will show how all these subalgebras appear naturally in terms of derivations of the Albert algebra, the first usual model of F_4. Afterwards, Section 3 will deal with the general properties of the gradings related to these subalgebras, explaining also the relationship to the finite order automorphisms. The next step, in Section 4, will be the description of the models and the gradings. This section is divided in subsections according to the associated grading groups: $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_2^2$ and $\mathbb{Z}_2^3$.

Finally, the last section will be devoted to some general remarks about relationships and applications: i) the models of the remaining exceptional Lie algebras suggested by this work, particularly E_8; ii) the unified models through series and magic squares; and, as regards our gradings of F_4, iii) all our gradings are toroidal, iv) there is only one grading of F_4 by a root system (C_3), v) what modules appear in the decompositions, vi) which gradings are refinements of the others, and vii) the relation between the chain of subalgebras and the inverted chain of subgroups.

2 Maximal rank semisimple subalgebras of F_4

Our first objective is the description of the subalgebras in which our models will be based on. Although they will be the semisimple subalgebras of rank 4, the arguments will be valid in the more general situation of reductive subalgebras. For algebraically closed fields of characteristic zero there is an easy method of determining the maximal rank reductive subalgebras.

2.1 Notice first that the notion of reductive subalgebra is associated to that of closed and symmetric subset [Bo, Ch VI, §1.7]:

If $\mathcal{L} = \sum_{\alpha\in\Phi\cup\{0\}} L_\alpha$ is the root decomposition relative to a Cartan subalgebra H $(= L_0)$ of a simple Lie algebra $\mathcal{L}$, a subset Γ of the root system ϕ is said to be *closed* if $\alpha, \beta \in \Gamma$ such that $\alpha + \beta \in \Phi$ implies $\alpha + \beta \in \Gamma$, and it is said to be *symmetric* if $-\Gamma = \Gamma$. In this situation the subalgebra $\mathfrak{h} := \sum_{\alpha\in\Gamma\cup\{0\}} L_\alpha$ is reductive, $\mathfrak{h} = Z(\mathfrak{h}) \oplus [\mathfrak{h}, \mathfrak{h}]$ with $Z(\mathfrak{h}) \subset H$ and $\mathcal{L}$ is an $\mathfrak{h}$-completely reducible module.

Conversely, if $\mathfrak{h}$ is a reductive subalgebra of $\mathcal{L}$ of maximal rank, $\mathfrak{h}$ is the sum of its center $Z(\mathfrak{h})$ and its semisimple part $[\mathfrak{h}, \mathfrak{h}]$, hence a Cartan subalgebra of $[\mathfrak{h}, \mathfrak{h}]$ plus $Z(\mathfrak{h})$ is a Cartan subalgebra of $\mathcal{L}$, since its dimension is the rank of $\mathcal{L}$ and it acts in a semisimple way. The root system Φ' of $[\mathfrak{h}, \mathfrak{h}]$ relative to such Cartan subalgebra can be considered to be contained in the root system Φ of $\mathcal{L}$. Now, for any $\alpha, \beta \in \Phi'$ such that $\alpha + \beta \in \Phi$, $L_{\alpha+\beta} = [L_\alpha, L_\beta] \subset [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, so Φ' is closed (clearly symmetric since being a root system).

2.2 The maximal closed and symmetric subsets of Φ are classified in [Bo, Ch VI, §4, exercise 4]:

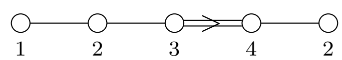
Let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ be a basis of Φ, let $\tilde{\alpha} = \sum n_i\alpha_i$ be the maximal root, let us fix $i \in \{1, \ldots, l\}$.

a) The closed and symmetric subset $\Phi_i = \{\alpha \in \Phi \mid \alpha = \sum_{j\neq i} k_j\alpha_j\}$ is maximal if and only if $n_i = 1$. In any case the subalgebra $\mathfrak{h} := \sum_{\alpha\in\Phi_i\cup\{0\}} L_\alpha$ has a one-dimensional center and the Dynkin diagram of its semisimple part is obtained by removing the node i of the Dynkin diagram of $\mathcal{L}$.

b) In the case $n_i > 1$, the set $\Gamma_i = \{\alpha \in \Phi \mid \alpha = \sum_{j=1}^{l} k_j\alpha_j, k_i \equiv 0 \pmod{n_i}\}$ is also closed and symmetric, and it is maximal if and only if n_i is prime. Moreover, $\{-\tilde{\alpha}, \alpha_j \mid j \neq i\}$ is a basis of Γ_i. Hence the subalgebra $\mathfrak{h}$ with root system Γ_i is semisimple of maximal rank and its Dynkin diagram is obtained by removing the node i of the affine Dynkin diagram. Such extended Dynkin diagrams appear in the table Aff1 in [Ka, p. 54], and independently in [F], obtained by joining the vertices i and j ($l+1$ nodes with $\alpha_0 = -\tilde{\alpha}$) by means of $|\langle\alpha_i, \alpha_j\rangle|$ edges.

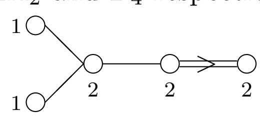
Every maximal closed and symmetric subset of Φ is transformed by an element of the Weyl group into one of the subsets described above. The corresponding subalgebras are conjugated by means of $\operatorname{Aut}\mathcal{L}$. Now, a closed and symmetric subset of Φ different than Φ is contained in some maximal subset, so it can be obtained iterating the previous procedure.

2.3 Applying the above in order to know the semisimple subalgebras of F_4 of rank 4, let us take the affine Dynkin diagram of F_4,



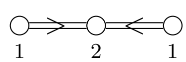
$F_4^{(1)}$

and let us remove the nodes corresponding to α_1, α_2 and α_4, thus obtaining the maximal subalgebras of types $A_1 \oplus C_3$, $2A_2$ and B_4 respectively. Now, the affine Dynkin diagram of B_4 is



$B_4^{(1)}$

so, by removing the three nodes marked with 2 we obtain $2A_1 \oplus C_2$, $A_3 \oplus A_1$ and D_4. Lastly we get $2A_1$ from the affine Dynkin diagram of C_2



$C_2^{(1)}$

when we remove the node marked with 2. Any other choice leads to the same subalgebras. In conclusion,

Theorem. *There are seven maximal rank semisimple subalgebras of F_4, namely: B_4, $A_3 \oplus A_1$, $2A_2$, $C_3 \oplus A_1$, D_4, $C_2 \oplus 2A_1$ and $4A_1$.*

2.4 We have recalled the diagram-based method because it is closely related to the grading group, as we will see in the next section, but in fact all the reductive subalgebras of F_4 appear in [Dy, table 11], in which Dynkin classifies the regular semisimple subalgebras of the exceptional complex simple Lie algebras. A subalgebra $\mathfrak{h}$ of a simple Lie algebra $\mathcal{L}$ is regular if there exists a basis consisting of elements of some Cartan subalgebra H of the algebra $\mathcal{L}$ and root vectors of the algebra $\mathcal{L}$ relative to H. It is clear that a reductive subalgebra of maximal rank is regular, and it can be obtained from the table 11 by adding the corresponding centers. Besides they are unique up to conjugation, according to [Dy, p. 148].

2.5 Although our main theorem about gradings will be stated for algebraically closed fields of characteristic zero, the resulting models and gradings will not need such restrictions on the field. Thus, we are going to describe our subalgebras in terms of the derivations of the Albert algebra, requiring only that the characteristic of the field F (not necessarily algebraically closed) is different than 2 and 3.

We will use the notations of [S, Ch IV, §1-2]. Let $(\mathcal{C}, n)$ be any Cayley algebra over F, that is, $\mathcal{C}$ an eight-dimensional unital composition algebra and n its quadratic form. Let J be the exceptional central simple Jordan algebra $J = H_3(\mathcal{C}) = \{x \in \mathrm{Mat}_{3\times 3}(\mathcal{C}) \mid \bar{x} = x^t\}$, formed by the selfdual elements with respect to the standard involution,

$$x = \begin{pmatrix} \xi_1 & c & \bar{b} \\ \bar{c} & \xi_2 & a \\ b & \bar{a} & \xi_3 \end{pmatrix} \equiv \xi_1 e_1 + \xi_2 e_2 + \xi_3 e_3 + a_1 + b_2 + c_3$$

for any $\xi_i \in F$, $a, b, c \in \mathcal{C}$. Thus, e_1, e_2 and e_3 are pairwise orthogonal idempotents, and we have the Peirce decomposition $J = \sum_{i \le j} J_{ij}$, where $J_{ij} = \{x \in J \mid xe_i = xe_j = \frac{1}{2}x\} = \{a_k \mid a \in \mathcal{C}\}$ if $i \neq j \neq k$. If B is a vector subspace of $\mathcal{C}$, we will denote by $B_k \equiv \{b_k \mid b \in B\}$ the subspace of J, so $J_{ij} = \mathcal{C}_k$. Take

$$\begin{array}{rccl} <\ ,\ >\colon & J \times J & \to & F \\ & (x, y) & \mapsto & <x, y> = t(xy), \end{array}$$

which is a nondegenerate symmetric bilinear form in J, $\mathrm{Der}\, J$-invariant.

Let $\mathcal{K} \le \mathcal{Q} \le \mathcal{C}$ be a chain of composition subalgebras of $\mathcal{C}$ of dimensions 2, 4 and 8 respectively, and the corresponding chain of Jordan algebras $H_3(\mathcal{K}) \le H_3(\mathcal{Q}) \le H_3(\mathcal{C})$. Their derivation algebras are, respectively, $\mathrm{Der}\, H_3(\mathcal{K}) \cong A_2$, $\mathrm{Der}\, H_3(\mathcal{Q}) \cong C_3$ and $\mathrm{Der}\, H_3(\mathcal{C}) \cong F_4$ (see [JJa] and [ChS], or the first row in the magic square [S, p. 122]). Looking at the second column of the magic square [S, p. 122], $\mathrm{Der}\, H_3(\mathcal{K})$ can be constructed as $\mathrm{Der}\, H_3(F) \oplus \mathrm{Der}\, K \oplus H_3(F)_0 \otimes K_0$, and similarly $\mathrm{Der}\, H_3(\mathcal{Q})$ and $\mathrm{Der}\, H_3(\mathcal{C})$ (replacing the composition algebra), so that we have another chain $\mathrm{Der}\, H_3(\mathcal{K}) \le \mathrm{Der}\, H_3(\mathcal{Q}) \le \mathrm{Der}\, H_3(\mathcal{C})$ (since $\mathrm{Der}\, K \le \mathrm{Der}\, \mathcal{Q} \le \mathrm{Der}\, \mathcal{C}$, [BeDrE2]). Another sequence arises taking $0 \neq a \in \mathcal{K}$ with $t(a) = 0$, and the map

$$\begin{array}{rccl} \sigma\colon & \mathcal{C} \times \mathcal{C} & \to & \mathcal{K} \\ & (x, y) & \mapsto & \sigma(x, y) = n(a)n(x, y) - n(ax, y)a, \end{array}$$

which is a nondegenerate hermitian form and, up to a scalar, does not depend on a. The Lie algebra of derivations $\mathrm{Der}\, \mathcal{C}$ is a central simple Lie algebra of type G_2, and their subalgebras $\{d \in \mathrm{Der}\, \mathcal{C} \mid d(\mathcal{Q}) = 0\}$ and $\{d \in \mathrm{Der}\, \mathcal{C} \mid d(\mathcal{K}) = 0\}$ can be seen as the special unitary Lie algebras $\mathrm{su}(\mathcal{Q}^\perp, \sigma)$ and $\mathrm{su}(\mathcal{K}^\perp, \sigma)$ respectively, of types A_1 and A_2 ([Dr],[BeDrE2]) since the restrictions $\sigma|_{\mathcal{Q}^\perp \times \mathcal{Q}^\perp}$ and $\sigma|_{\mathcal{K}^\perp \times \mathcal{K}^\perp}$ are also nondegenerate.

Now, several semisimple Lie subalgebras of rank 4 of F_4 arise in a natural way:

Proposition 1. *With the notations above, there are the following isomorphisms of Lie algebras:*

i) $\mathfrak{h}_0 = \{d \in \mathrm{Der}\, J \mid d(e_i) = 0 \quad \forall i\} \approx \mathrm{o}(\mathcal{C}, n)$, *of type* D_4,

ii) $\mathfrak{h}_1 = \{d \in \mathrm{Der}\, J \mid d(e_1) = 0\} \approx \mathrm{o}(\mathcal{C}_1 \oplus F(e_2 - e_3), <\ ,\ >)$, *of type* B_4,

iii) $\mathfrak{h}_2 = \{d \in \mathrm{Der}\, J \mid d(H_3(\mathcal{Q})) \subset H_3(\mathcal{Q})\} \approx \mathrm{su}(\mathcal{Q}^\perp, \sigma) \oplus \mathrm{Der}\, H_3(\mathcal{Q})$, *of type* $A_1 \oplus C_3$,

iv) $\mathfrak{h}_3 = \{d \in \mathrm{Der}\, J \mid d(H_3(\mathcal{K})) \subset H_3(\mathcal{K})\} \approx \mathrm{su}(\mathcal{K}^\perp, \sigma) \oplus \mathrm{Der}\, H_3(\mathcal{K})$, *of type* $2A_2$,

v) $\mathfrak{h}_4 = \{d \in \mathrm{Der}\, J \mid d(\mathcal{K}_1^\perp) \subset \mathcal{K}_1^\perp\} \approx \mathrm{o}(\mathcal{K}^\perp, n) \oplus \mathrm{o}(\mathcal{K}_1 \oplus F(e_2 - e_3), <\ ,\ >)$, *of type* $A_3 \oplus A_1$,

vi) $\mathfrak{h}_5 = \{d \in \mathrm{Der}\, J \mid d(\mathcal{Q}_1^\perp) \subset \mathcal{Q}_1^\perp\} \approx \mathrm{o}(\mathcal{Q}^\perp, n) \oplus \mathrm{o}(\mathcal{Q}_1 \oplus F(e_2 - e_3), <\ ,\ >)$, *of type* $2A_1 \oplus C_2$,

vii) $\mathfrak{h}_6 = \{d \in \mathrm{Der}\, J \mid d(H_3(\mathcal{Q})) \subset H_3(\mathcal{Q}),\ d(e_i) = 0 \quad \forall i\} \approx \mathrm{o}(\mathcal{Q}, n) \oplus \mathrm{o}(\mathcal{Q}^\perp, n)$, *of type* $4A_1$.

Moreover, all the pairs $(F_4, \mathfrak{h}_i)$ *are reductive, that is, there are* $\mathfrak{m}_i$ *subspaces of* F_4 *such that* $F_4 = \mathfrak{h}_i \oplus \mathfrak{m}_i$ *and* $[\mathfrak{h}_i, \mathfrak{m}_i] \subset \mathfrak{m}_i$*.*

Proof. i) The first isomorphism is known [S, Ch IV, proof of th. 4.9], but we will do a sketch to use the notations during the rest of the proof.

If $U \in \mathrm{o}(\mathcal{C}, n)$, by the principle of local triality there are unique maps $U', U'' \in \mathrm{o}(\mathcal{C}, n)$ satisfying $U(xy) = U'(x)y + xU''(y)$ for any $x, y \in \mathcal{C}$. Take $S\colon \mathcal{C} \to \mathcal{C}$ the standard involution $S(x) = \bar{x}$. Now, we define

$$D_U\colon J \to J \quad \text{by} \begin{cases} D_U(e_i) = 0 \\ D_U(a_i) = U_i(a)_i \end{cases}$$

for

$$U_1 = S^{-1}US, \qquad U_2 = U', \qquad U_3 = U''.$$

Thus the set $\mathfrak{h}_0 = \{d \in \mathrm{Der}\,\mathcal{C} \mid d(e_i) = 0\ \forall i\} = \{D_U \mid U \in \mathrm{o}(\mathcal{C}, n)\}$ is isomorphic to $\mathrm{o}(\mathcal{C}, n)$ since $[D_U, D_T] = D_{[U,T]}$.

Moreover, if for any $i \in \{1, 2, 3\}$ we denote $d_{a/i} \equiv [R_{e_j - e_k}, R_{a_i}] \in \mathrm{Der}\,J$ (i, j, k cyclic permutation of $1, 2, 3$) and $\mathcal{D}_i \equiv \{d_{a/i} \mid a \in \mathcal{C}\}$, then $\mathrm{Der}\,J = \mathcal{D}_0 \oplus \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \mathcal{D}_3$, and, since $[D_U, d_{a/i}] = d_{U_i(a)/i}$, $\mathcal{D}_i$ are $\mathfrak{h}_0$-modules and $\mathfrak{m}_0 = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \mathcal{D}_3$ is a complement $\mathfrak{h}_0$-invariant.

ii) The isomorphism $\mathfrak{h}_1 \approx B_4$ was already mentioned in the introduction (see also [Ja4] and [Ja2]). Anyway, it is easy to check it directly at the same time than the cases $\mathfrak{h}_4$ and $\mathfrak{h}_5$.

Notice first that $\mathfrak{h}_1 = \{d \in \mathrm{Der}\,\mathcal{C} \mid d(e_1) = 0\} = \mathcal{D}_0 \oplus \mathcal{D}_1$, since

$$\begin{array}{ll} d_{a/i}(e_i) = 0, & d_{a/i}(b_i) = \frac{1}{2}n(a, b)(e_j - e_k) \\ d_{a/i}(e_j) = -\frac{1}{2}a_i, & d_{a/i}(b_j) = a_i b_j \\ d_{a/i}(e_k) = \frac{1}{2}a_i, & d_{a/i}(b_k) = -a_i b_k, \end{array}$$

and that $\mathfrak{m}_1 = \mathcal{D}_2 \oplus \mathcal{D}_3$ is an $\mathfrak{h}_1$-invariant complement.

Now, if X is a composition subalgebra of $\mathcal{C}$, it holds that $X_1^\perp \oplus F(e_2 + e_3)$ is a subalgebra of J and the set of derivations of J which leave it invariant is the subalgebra $\mathfrak{h}_X \equiv \{d \in \mathrm{Der}\,J \mid d(X_1^\perp) \subset X_1^\perp\} \subset \mathfrak{h}_1 = \mathcal{D}_0 \oplus \mathcal{D}_1$.

An element D_U belongs to $\mathfrak{h}_X$ iff $(U_1(X^\perp))_1 \subset X_1^\perp$, or equivalently, if $U(X^\perp) \subset X^\perp$. If we denote $\varphi_{a,b} \equiv n(a, -)b - n(b, -)a$ for any $a, b \in \mathcal{C}$, the set $\mathrm{o}(\mathcal{C}, n)$ is equal to $\varphi_{\mathcal{C},\mathcal{C}} = \varphi_{X,X} \oplus \varphi_{X,X^\perp} \oplus \varphi_{X^\perp,X^\perp}$, with $\varphi_{X,X}(X^\perp) = 0$, $\varphi_{X^\perp,X}(X^\perp) \subset X$ and $\varphi_{X^\perp,X^\perp}(X^\perp) \subset X^\perp$, so that $\{U \in \mathrm{o}(\mathcal{C}, n) \mid U(X^\perp) \subset X^\perp\} = \varphi_{X,X} \oplus \varphi_{X^\perp,X^\perp}$.

On the other hand, $d_{a/1}(X_1^\perp) \subset n(a, X^\perp)(e_2 - e_3)$, so that an element $d_{a/1}$ belongs to $\mathfrak{h}_X$ iff $a \in X^{\perp\perp} = X$.

Hence, $\mathfrak{h}_X = D_{\varphi_{X^\perp,X^\perp}} \oplus (D_{\varphi_{X,X}} \oplus d_{X/1})$, where

$$D_{\varphi_{X^\perp,X^\perp}} = \{d \in \mathfrak{h}_X \mid d(X_1) = 0\}$$

and

$$D_{\varphi_{X,X}} \oplus d_{X/1} = \{d \in \mathfrak{h}_X \mid d(X_1^\perp) = 0\} \tag{2}$$

are obviously two ideals.

The first ideal is isomorphic to $\varphi_{X^\perp,X^\perp} \approx \mathrm{o}(X^\perp, n)$, a Lie algebra of the type $\mathrm{o}(7)$, $\mathrm{o}(6)$, $\mathrm{o}(4)$ and 0 (B_3, $D_3 \approx A_3$, $D_2 \approx 2A_1$ and 0) if X is the composition algebra F, $\mathcal{K}$, $\mathcal{Q}$ and $\mathcal{C}$ respectively.

Since $d_{X/1}(X_1) \subset F(e_2 - e_3)$, $d_{X/1}(e_2 - e_3) \subset X_1$, $d_{\varphi_{X,X}}(X_1) = X_1$ and $d_{\varphi_{X,X}}(e_2 - e_3) = 0$, we can take the map

$$\begin{array}{rcl} \{d \in \operatorname{Der} J \mid d(X_1^\perp) = 0\} & \to & \mathrm{o}(X_1 \oplus F(e_2 - e_3), <\, ,\, >) \\ d & \mapsto & d|_{X_1 \oplus F(e_2 - e_3)} \end{array}$$

which is well defined ($<\, ,\, >$ is $\operatorname{Der} J$-invariant and $<\, ,\, > |_{X_1 \oplus F(e_2 - e_3)}$ is nondegenerate too), and it is a monomorphism between vector spaces of the same dimension $\binom{1+\dim X}{2}$. So the ideal in (2) is isomorphic to $\mathrm{o}(X_1 \oplus F(e_2 - e_3), <\, ,\, >)$, a Lie algebra of the type $\mathrm{o}(1)$, $\mathrm{o}(3)$, $\mathrm{o}(5)$ and $\mathrm{o}(9)$ respectively (F, $B_1 \approx A_1$, $B_2 \approx C_2$ and B_4). Therefore $\mathfrak{h}_F \approx B_3 \oplus F$, $\mathfrak{h}_4 = \mathfrak{h}_{\mathcal{K}} \approx A_3 \oplus A_1$, $\mathfrak{h}_5 = \mathfrak{h}_{\mathcal{Q}} \approx 2A_1 \oplus C_2$ and $\mathfrak{h}_1 = \mathfrak{h}_{\mathcal{C}} \approx B_4$ are reductive subalgebras of F_4 of rank 4 (the first one is not semisimple) whose corresponding invariant complements are $\mathfrak{m}_X = D_{\varphi_{X,X^\perp}} \oplus d_{X^\perp/1} \oplus \mathcal{D}_2 \oplus \mathcal{D}_3$.

iii) Let us take now $\mathfrak{h}_{(X)} \equiv \{d \in \operatorname{Der} J \mid d(H_3(X)) \subset H_3(X)\}$ and the Lie algebras homomorphism

$$\begin{array}{rcl} \Phi_X \colon \mathfrak{h}_{(X)} & \to & \operatorname{Der} H_3(X) \\ d & \mapsto & d|_{H_3(X)} \end{array}$$

whose kernel is $\operatorname{Ker} \Phi_X = \{d \in \operatorname{Der} J \mid d(H_3(X)) = 0\} \subset \mathfrak{h}_0$.

Recall from [S, p. 81] that $\mathrm{o}(\mathcal{C}, n) = \operatorname{Der} \mathcal{C} \oplus R_{\mathcal{C}_0} \oplus L_{\mathcal{C}_0}$, where L_a and R_a denote the left and right multiplications by the element $a \in \mathcal{C}_0$. Let $U = d + L_a + R_b$ be an arbitrary element in $\mathrm{o}(\mathcal{C}, n)$ ($d \in \operatorname{Der} \mathcal{C}$, $a, b \in \mathcal{C}_0$). Then $D_U \in \operatorname{Ker} \Phi_X$ iff $U_i(X) = 0$ for all i. But $d = \frac{1}{3}(U_1 + U_2 + U_3)$, so $d(X) = 0$ and hence the maps $L_a + R_b = U - d$ and $-R_a + R_b + L_b = U' - d$ annihilate X, in particular annihilate $1 \in X$, so that $a + b = -a + 2b = 0$ and $a = b = 0$. That is, the kernel $\operatorname{Ker} \Phi_X = \{D_U \mid U \in \operatorname{Der} \mathcal{C}, U(X) = 0\} \approx \{U \in \operatorname{Der} \mathcal{C} \mid U(X) = 0\}$ is isomorphic to G_2, A_2, A_1 and 0 if X is equal to F, $\mathcal{K}$, $\mathcal{Q}$ and $\mathcal{C}$ respectively, and so of dimension 14, 8, 3 and 0.

It is easy to see that $\mathfrak{h}_{(X)} = D_{\{d \in \operatorname{Der} \mathcal{C} | d(X) \subset X\} \oplus R_{X_0} \oplus L_{X_0}} \oplus (\oplus_i d_{X/i})$ and that an invariant complement is $\mathfrak{m}_{(X)} = D_{D_{X_0, X^\perp} \oplus R_{X^\perp} \oplus L_{X^\perp}} \oplus (\oplus_i d_{X^\perp/i})$, with the usual notation $D_{a,b} = [R_a, R_b] + [L_a, R_b] + [L_a, L_b] \in \operatorname{Der} \mathcal{C}$ used in [S, p. 77]. As $\dim\{d \in \operatorname{Der} \mathcal{C} \mid d(X) \subset X\} = 14, 8, 6, 14$ by [Dr] or [BeDrE2], the dimension of $\mathfrak{h}_{(X)}$ is equal to $\dim\{d \in \operatorname{Der} \mathcal{C} \mid d(X) \subset X\} + 5 \dim X - 2 =$ 17, 16, 24 and 52 respectively, the same than $\dim \operatorname{Ker} \Phi_X + \dim \operatorname{Der} H_3(X)$, since $\operatorname{Der} H_3(X)$ is isomorphic to A_1, A_2, C_3 and F_4 respectively. Therefore Φ_X is an epimorphism, and $\mathfrak{h}_{(X)} \approx \operatorname{Ker} \Phi_X \oplus \operatorname{Im} \Phi_X = \operatorname{Ker} \Phi_X \oplus \operatorname{Der} H_3(X)$ is of type $G_2 \oplus A_1$, $2A_2$, $A_1 \oplus C_3$ and F_4 respectively, with $\mathfrak{h}_{\mathcal{K}} = \mathfrak{h}_3$ and $\mathfrak{h}_{\mathcal{Q}} = \mathfrak{h}_2$.

iv) Finally, the subalgebra $\mathfrak{h}_6 = \mathfrak{h}_0 \cap \mathfrak{h}_2 = \{d \in \operatorname{Der} J \mid d(H_3(\mathcal{Q})) \subset H_3(\mathcal{Q}),\, d(e_i) = 0\, \forall i\} = D_{\{d \in \operatorname{Der} \mathcal{C} | d(\mathcal{Q}) \subset \mathcal{Q}\} \oplus R_{\mathcal{Q}_0} \oplus L_{\mathcal{Q}_0}} = D_{\varphi_{\mathcal{Q},\mathcal{Q}} \oplus \varphi_{\mathcal{Q}^\perp, \mathcal{Q}^\perp}} \approx \varphi_{\mathcal{Q},\mathcal{Q}} \oplus \varphi_{\mathcal{Q}^\perp, \mathcal{Q}^\perp} \approx \mathrm{o}(\mathcal{Q}, n) \oplus \mathrm{o}(\mathcal{Q}^\perp, n)$ is of type $2\,\mathrm{o}(4)$ or $4A_1$, and it has an invariant complement $\mathfrak{m}_6 = D_{\varphi_{\mathcal{Q},\mathcal{Q}^\perp}} \oplus \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \mathcal{D}_3$. □

Remark. In order to obtain a new model of an algebra $\mathcal{L}$, one can start with a description of the algebra, then identifying different pieces whose direct sum is $\mathcal{L}$ to models and finally translating the bracket in $\mathcal{L}$ through these bijections. In this kind of models, it is very useful to know a complete description of the pieces. This is why we have added a description of the complements $\mathfrak{m}$ in the proof, which could also be used to construct nonsplit Lie algebras of type F_4, analogously to the models of G_2 obtained in [BeDrE1]. Furthermore, they provide a collection of examples of Lie-Yamaguti algebras with standard embedding F_4 (consult [BeDrE2] for more information about this topic). However, the description of the complements $\mathfrak{m}$ is not essential to the constructions in this work, because their Lie algebra structures will be proved by checking directly the Jacobi identity, instead of using isomorphisms.

3 Gradings based on reductive subalgebras of maximal rank.

3.1 Let us consider again an algebraically closed field F of characteristic zero. The next theorem explains how any reductive subalgebra $\mathfrak{h}$ of maximal rank of a simple Lie algebra $\mathcal{L}$ provides a grading of $\mathcal{L}$ such that the identity component is $\mathfrak{h}$ and verifying some remarkable properties which will help us to recover the bracket in $\mathcal{L}$.

Theorem 1. *Let $\mathcal{L}$ be a (finite-dimensional) simple Lie algebra, $\mathcal{L} = \sum_{\alpha\in\Phi\cup\{0\}} L_\alpha$ the root decomposition relative to a Cartan subalgebra H of $\mathcal{L}$, and Φ' a closed and symmetric subset of the root system Φ. Let G be the abelian group $\mathbb{Z}\Phi/\mathbb{Z}\Phi'$. Then:*

a) $\Phi \cap \mathbb{Z}\Phi' = \Phi'$

b) $\mathcal{L} = \oplus_{g\in G}\mathcal{L}_g$ *is G-graded, being $\mathcal{L}_0 = \mathfrak{h} = H \oplus \sum_{\alpha\in\Phi'} L_\alpha$ a reductive subalgebra and for any $0 \neq g \in G$ either $\mathcal{L}_g = 0$ or $\mathcal{L}_g$ is an $\mathfrak{h}$-irreducible module. Besides G is generated by the set $\{g \in G \mid \mathcal{L}_g \neq 0\}$.*

c) *If $\mathcal{L} = \oplus_{g\in\tilde{G}} M_g$ is another grading by an abelian group $\tilde{G}$, where $M_0 \supset \mathfrak{h}$ and $\tilde{G}$ is generated by $\{g \in \tilde{G} \mid M_g \neq 0\}$, then there is a group epimorphism $\pi\colon G \to \tilde{G}$ such that $\mathcal{L}_g \subset M_{\pi(g)}$ for any $g \in G$.*

d) *If $g_1, g_2 \in G \setminus \{0\}$, $g_1 + g_2 \neq 0$, $\mathrm{Hom}_{\mathfrak{h}}(\mathcal{L}_{g_1} \otimes \mathcal{L}_{g_2}, \mathcal{L}_{g_1+g_2})$ is generated by the restriction of the Lie bracket $[\ ,\ ]\colon \mathcal{L}_{g_1} \otimes \mathcal{L}_{g_2} \to \mathcal{L}_{g_1+g_2}$.*

e) *If $g_1, g_2 \in G$, $g_1 + g_2 \neq 0$, then $[\mathcal{L}_{g_1}, \mathcal{L}_{g_2}] = \mathcal{L}_{g_1+g_2}$.*

f) *$\mathfrak{h}$ is semisimple if and only if G is a finite group.*

g) *$\mathcal{L}_g \neq 0$ for all $g \in G$ if and only if the bracket of any two irreducible components of $\mathcal{L}$ not contained in $\mathfrak{h}$ is not zero. In this case $\mathfrak{h}$ is semisimple.*

Proof. **a)** Given $\alpha_1, \dots, \alpha_n \in \Phi'$ such that $\alpha = \alpha_1 + \cdots + \alpha_n \in \Phi$, let us show that $\alpha \in \Phi'$ by induction on n (it is trivial in case $n = 1$). As $(\alpha, \alpha) > 0$, there is an index $i \in \{1, \dots, n\}$ such that $(\alpha, \alpha_i) > 0$, so $\alpha - \alpha_i = \alpha_1 + \cdots + \hat{\alpha}_i + \cdots + \alpha_n \in \Phi$ (the hat means that the marked summand doesn't appear). By the induction hypothesis $\alpha - \alpha_i \in \Phi'$, and so $\alpha = \alpha_i + (\alpha - \alpha_i) \in (\Phi' + \Phi') \cap \Phi \subset \Phi'$.

b) It is clear that $\mathcal{L}$ is graded by the abelian group $G = \mathbb{Z}\Phi/\mathbb{Z}\Phi'$, with $L_\alpha \subset \mathcal{L}_{\alpha+\mathbb{Z}\Phi'}$; and, since $\{\alpha + \mathbb{Z}\Phi' \mid \alpha \in \Phi\}$ is a set of generators of G contained in $\{g \in G \mid \mathcal{L}_g \neq 0\}$, this set also generates G.

Writing $\mathfrak{h} = \sum_{\alpha\in\Phi'\cup\{0\}} L_\alpha$, it is a reductive Lie subalgebra of $\mathcal{L}$ (2.1), obviously contained in $\mathcal{L}_0 = \sum_{\alpha+\mathbb{Z}\Phi'=\mathbb{Z}\Phi'} L_\alpha$. So $\mathfrak{h} = \mathcal{L}_0$ holds, by a).

By complete reducibility, we can decompose $\mathcal{L} = \mathfrak{h} \oplus \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_r$, where $\mathfrak{m}_i$ are $\mathfrak{h}$-irreducible modules. Since $\mathcal{L} = H \oplus \sum_{\alpha\in\Phi} L_\alpha = \mathfrak{h} \oplus \sum_{\alpha\in\Phi\setminus\Phi'} L_\alpha$, and $\mathfrak{m}_i$ are $\mathfrak{h}$-invariant, there are $\Phi_1, \dots, \Phi_r$ nonempty subsets of Φ such that $\Phi = \Phi' \dot\cup \Phi_1 \dot\cup \cdots \dot\cup \Phi_r$ and $\mathfrak{m}_i = \sum_{\alpha\in\Phi_i} L_\alpha$.

Let us see now that for any $i \in \{1, \dots, r\}$ there exists $0 \neq g \in G$ such that $\mathfrak{m}_i = \mathcal{L}_g$. Thus, each of the nonzero homogeneous components of the grading will be some of the irreducible submodules $\mathfrak{m}_i$. To do that, let us take μ_i a maximal root in Φ_i. Since $\mathfrak{m}_i$ is $[\mathfrak{h}, \mathfrak{h}]$-irreducible, $\mathfrak{m}_i = U([\mathfrak{h}, \mathfrak{h}])^- L_{\mu_i}$ and the whole set Φ_i is contained in $\mu_i + \mathbb{Z}\Phi'$ (element in G). Hence $\mathfrak{m}_i \subset \mathcal{L}_{\mu_i+\mathbb{Z}\Phi'}$.

To prove the equality, let us suppose that there exists $\eta_i \in \Phi_i$ and $\eta_j \in \Phi_j$ $(i \neq j)$ such that $\eta_i - \eta_j \in \mathbb{Z}\Phi'$. So there are $\alpha_1, \ldots, \alpha_n \in \Phi'$ such that $\eta_i - \eta_j = \alpha_1 + \cdots + \alpha_n$, and we can choose η_i and η_j such that n is minimum.

As $(\eta_i, \eta_j + \alpha_1 + \cdots + \alpha_n) = (\eta_i, \eta_i) > 0$, it follows that either there is an index $l \in \{1, \ldots, n\}$ such that $(\eta_i, \alpha_l) > 0$ or $(\eta_i, \eta_j) > 0$. In the first case, $\eta_i - \alpha_l \in \Phi$, so that $L_{\eta_i - \alpha_l} = [L_{\eta_i}, L_{-\alpha_l}] \subset [\mathfrak{m}_i, \mathfrak{h}] \subset \mathfrak{m}_i$ and $\eta_i' = \eta_i - \alpha_l \in \Phi_i$ with $\eta_i' - \eta_j = \alpha_1 + \cdots + \hat{\alpha}_l + \cdots + \alpha_n$, a contradiction with the choice of n. In the second case, $\eta_i - \eta_j \in \Phi$ ($\eta_i \neq \eta_j$, since $\Phi_i \cap \Phi_j = \emptyset$, and $\eta_i \neq -\eta_j$, since $(\eta_i, -\eta_i) < 0$), it follows that $\eta_i - \eta_j \in \Phi' = \Phi \cap \mathbb{Z}\Phi'$ by a), and then $L_{\eta_i} = [L_{\eta_j}, L_{\eta_i - \eta_j}] \subset [\mathfrak{m}_j, \mathfrak{h}] \subset \mathfrak{m}_j$, again an absurd. In consequence for all $\eta_j \in \Phi_j$, $\eta_j + \mathbb{Z}\Phi' \neq \mu_i + \mathbb{Z}\Phi'$ and $\mathcal{L}_{\mu_i + \mathbb{Z}\Phi'} = \sum_{\beta \in \Phi, \beta + \mathbb{Z}\Phi' = \mu_i + \mathbb{Z}\Phi'} L_\beta = \mathfrak{m}_i$.

c) Since $\mathfrak{h} \subset M_0$, it is clear that any root space is homogeneous, hence if Δ is a basis of Φ, we can take $\pi\colon \Delta \to \tilde{G}$ such that $L_\alpha \subset M_{\pi(\alpha)}$. There exists an unique extension to the free $\mathbb{Z}$-module generated by Δ such that $\pi\colon \mathbb{Z}\Delta = \mathbb{Z}\Phi \to \tilde{G}$ is a $\mathbb{Z}$-modules homomorphism. Note that $L_\alpha \subset M_{\pi(\alpha)}$ for any $\alpha \in \Phi^+$ (analogously for $\alpha \in \Phi^-$), since there is a chain $\alpha = \alpha_1 + \cdots + \alpha_s$ with $\alpha_i \in \Delta$ such that $\alpha_1 + \cdots + \alpha_i \in \Phi$ for all i, and because of the $\tilde{G}$-grading we have $L_\alpha = [[[L_{\alpha_1}, L_{\alpha_2}], \ldots], L_{\alpha_s}] \subset M_{\pi(\alpha_1) + \pi(\alpha_2) + \cdots + \pi(\alpha_s) = \pi(\alpha)}$. We thus get $\pi(\Phi') = 0$, since $M_0 \supset \mathfrak{h} \supset L_\alpha$ for any $\alpha \in \Phi'$; and consequently $\mathbb{Z}\Phi' \subset \ker \pi$ and the induced group homomorphism $\bar{\pi}\colon \mathbb{Z}\Phi/\mathbb{Z}\Phi' \to \tilde{G}$ is an epimorphism (the set of generators $\{g \in \tilde{G} \mid M_g \neq 0\}$ is contained in $\operatorname{Im} \bar{\pi}$).

More precisely, notice that if the $\tilde{G}$-grading verifies exactly the property b), that is, $M_0 = \mathfrak{h}$ and the nonzero components M_g are $\mathfrak{h}$-irreducible modules, both the G and $\tilde{G}$-gradings are equivalent because two decompositions of $\mathcal{L}$ as sums of the subalgebra $\mathfrak{h}$ plus $\mathfrak{h}$-irreducible submodules must be equal since the $\mathfrak{h}$-irreducible modules $\{\mathfrak{m}_i \mid i = 1, \ldots, r\}$ are all different and not isomorphic to the simple ideals of $\mathfrak{h}$.

d) By the Jacobi identity of $\mathcal{L}$, the brackets $[\mathcal{L}_{g_1}, \mathcal{L}_{g_2}]$ are $\mathfrak{h}$-invariant, so trivially the map $[\ ,\ ]|_{\mathcal{L}_{g_1} \otimes \mathcal{L}_{g_2}} \in \operatorname{Hom}_{\mathfrak{h}}(\mathcal{L}_{g_1} \otimes \mathcal{L}_{g_2}, \mathcal{L}_{g_1+g_2})$.

On the other hand, let us take μ_1 and μ_2 the maximal weights in $\mathcal{L}_{g_1}$ and $\mathcal{L}_{g_2}$ respectively ($\mathfrak{h}$-irreducible modules). If v_{μ_1} is a maximal vector for $\mathcal{L}_{g_1}$ of weight μ_1 and $v_{\sigma(\mu_2)}$ is a minimal vector for $\mathcal{L}_{g_2}$ of weight $\sigma(\mu_2)$ (σ the element in the Weyl group such that $\sigma(\Phi'^+) = \Phi'^-$), the $\mathfrak{h}$-module $\mathcal{L}_{g_1} \otimes \mathcal{L}_{g_2}$ is generated by $v_{\mu_1} \otimes v_{\sigma(\mu_2)}$ and any $f \in \operatorname{Hom}_{\mathfrak{h}}(\mathcal{L}_{g_1} \otimes \mathcal{L}_{g_2}, \mathcal{L}_{g_1+g_2})$ is determined by the image $f(v_{\mu_1} \otimes v_{\sigma(\mu_2)})$, contained in $L_{\mu_1 + \sigma(\mu_2)}$, trivial or one-dimensional (just in case $\mu_1 + \sigma(\mu_2)$ is a root), hence $\dim \operatorname{Hom}_{\mathfrak{h}}(\mathcal{L}_{g_1} \otimes \mathcal{L}_{g_2}, \mathcal{L}_{g_1+g_2}) \leq 1$.

In the case this space is one-dimensional, if we take μ the maximal root in $\mathcal{L}_{g_1+g_2}$ ($V(\mu)$ is a summand of $V(\mu_1) \otimes V(\mu_2)$), it is known that there exists α a weight in $V(\mu_2)$ such that $\mu = \mu_1 + \alpha$, so that $L_\mu = [L_{\mu_1}, L_\alpha] \subset [\mathcal{L}_{g_1}, \mathcal{L}_{g_2}]$ and thus $[\mathcal{L}_{g_1}, \mathcal{L}_{g_2}] \neq 0$.

e) It is enough to check that $[\mathcal{L}_{g_1}, \mathcal{L}_{g_2}] \neq 0$ for any $g_1, g_2 \in G$ such that $g_1 + g_2 \neq 0$ and $\mathcal{L}_{g_1+g_2} \neq 0$, by the irreducibility of $\mathcal{L}_{g_1+g_2}$.

With the notations in d) (the other case, g_1 or $g_2 = 0$, is trivial), $\mathcal{L}_{\mu_1+\mu_2+\mathbb{Z}\Phi'} \neq 0$, $\Phi \cap (\mu_1 + \mu_2 + \mathbb{Z}\Phi') \neq \emptyset$ and there exist $\alpha_1, \ldots, \alpha_n \in \Phi'$ such that $\mu_1 + \mu_2 + \alpha_1 + \cdots + \alpha_n \in \Phi$ ($\mathbb{Z}\Phi' = \mathbb{Z}^+\Phi'$ because $-\Phi' = \Phi'$). Let us take n minimum verifying the above.

As before, $(\mu_1 + \mu_2 + \alpha_1 + \cdots + \alpha_n, \mu_1 + \mu_2 + \alpha_1 + \cdots + \alpha_n) > 0$, but there doesn't exist $l \in \{1, \ldots, n\}$ such that $(\mu_1 + \mu_2 + \alpha_1 + \cdots + \alpha_n, \alpha_l) > 0$ by the choice of n, so that it may be assumed that $(\mu_1 + \mu_2 + \alpha_1 + \cdots + \alpha_n, \mu_1) > 0$ (analogously in case μ_2), $\mu_2 + \alpha_1 + \cdots + \alpha_n \in \Phi$ and $0 \neq [L_{\mu_1}, L_{\mu_2+\alpha_1+\cdots+\alpha_n}] \subset [\mathcal{L}_{\mu_1+\mathbb{Z}\Phi'}, \mathcal{L}_{\mu_2+\mathbb{Z}\Phi'}]$.

f) Let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ be a basis of Φ, and $\Delta' = \{\gamma_1, \ldots, \gamma_m\}$ be a basis of Φ'. Obviously

$\Delta' \subset \mathbb{Z}\Delta$.

The algebra $\mathfrak{h}$ is semisimple if and only if $Z(\mathfrak{h}) = 0$, equivalently if the rank of its semisimple part is the rank of $\mathcal{L}$, $m = l$.

If $m = l$, $\Delta \subset \mathbb{Q}\Delta'$ and there exist $n_1, \dots, n_l$ such that $n_i\alpha_i \in \mathbb{Z}\Delta'$. It is clear that the cardinal of the set $\mathbb{Z}\Phi/\mathbb{Z}\Phi'$ is less than $\prod_{i=1}^{l} n_i$, finite.

Conversely, if $m < l$, $\Phi \not\subset \sum_{i=1}^{m} \mathbb{R}\gamma_i$ and there exists $\alpha \in \Phi$ such that $\alpha \notin \mathbb{Q}\Phi'$. Hence $\{n\alpha + \mathbb{Z}\Phi' \mid n \in \mathbb{Z}\}$ is a subset of G with infinite different classes.

g) The condition $[\mathfrak{m}_i, \mathfrak{m}_j] \neq 0$ for all i, j, is equivalent, by e), to the fact $\mathcal{L}_{g_1+g_2} \neq 0$ if $\mathcal{L}_{g_1}, \mathcal{L}_{g_2} \neq 0$ (notice that $\mathcal{L}_0 \neq 0$, and, in any case, $0 \neq [L_\alpha, L_{-\alpha}] \subset [\mathcal{L}_{\alpha+\mathbb{Z}\Phi'}, \mathcal{L}_{-\alpha+\mathbb{Z}\Phi'}]$). So that the set $\hat{G} = \{g \in G \mid \mathcal{L}_g \neq 0\}$ is a subgroup, and, as $\hat{G}$ generates G, $G = \hat{G}$.

Since $\mathcal{L}$ is finite dimensional, it follows that G is finite and $\mathfrak{h}$ is semisimple as in f). □

3.2 On account of the above theorem, if $\mathcal{L} = \mathfrak{h} \oplus \mathfrak{m}$ and $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_r$ is the decomposition of $\mathcal{L}$ as a sum of $\mathfrak{h}$-irreducible modules ($\mathfrak{h}$ reductive of maximal rank), all of these modules are not trivial and not isomorphic among them. For each pair of indices (i, j), the bracket $[\mathfrak{m}_i, \mathfrak{m}_j]$ is either 0, or contained in $\mathfrak{h}$ (only in the case $\mathfrak{m}_j \approx \mathfrak{m}_i^*$), or contained in exactly one module $\mathfrak{m}_k$. In the latter case the bracket is determined up to scalar by any nonzero $\mathfrak{h}$-homomorphism $\mathfrak{m}_i \otimes \mathfrak{m}_j \to \mathfrak{m}_k$.

Therefore, we are in a good situation to describe $(\mathcal{L}, [\ ,\ ])$ by means of the subalgebra $\mathfrak{h}$, the $\mathfrak{h}$-modules $\mathfrak{m}_i$, and the homomorphisms $\mathrm{Hom}_{\mathfrak{h}}(\mathfrak{m}_i \otimes \mathfrak{m}_j, \mathfrak{m}_k)$; specially if $\mathfrak{h}$ is semisimple because the abelian grading group is finite, concretely of cardinal $r + 1$ if any bracket between two different irreducible components is not zero, which often happens.

3.3 It is interesting to notice that if $\tilde{G}$ is a subgroup of G, $\mathfrak{h}_{\tilde{G}} \equiv \sum_{g \in \tilde{G}} \mathcal{L}_g$ is again a reductive subalgebra of maximal rank (the Cartan subalgebra is contained in $\mathcal{L}_0$) containing $\mathfrak{h}$, semisimple with root system $\Phi' \cup \{\Phi_i \mid \mu_i + \mathbb{Z}\Phi' \in \tilde{G}\}$ if $\mathfrak{h}$ is semisimple too. The grading group of the construction based on $\mathfrak{h}_{\tilde{G}}$ is of course $G/\tilde{G}$. The converse also holds, there is an inverted chain of subalgebras and subgroups.

3.4 *Relationship with $\mathbb{Z}$- and $\mathbb{Z}_m$-gradings.*

Taking into consideration that there is a chain $\Phi' = \Phi_0 \subset \Phi_1 \subset \cdots \subset \Phi_s = \Phi$ of closed and symmetric subsets such that each of them is maximal in the following one, we could have checked the truth of the theorem in an indirect way by applying several times the general theory of automorphisms and gradings to the maximal closed and symmetric subsets. Each step would provide a cyclic grading group (either $\mathbb{Z}$ or $\mathbb{Z}_p$ with p a prime) jointly with the irreducibility of the homogeneous components. In particular we would know each quotient group G_i/G_{i+1} of the corresponding chain $G = G_0 \supset G_1 \supset \cdots \supset G_s$ (as in 3.3), and hence approximately the group G (not exactly because of the possible repeated primes).

In the notations of 2.2 and [OV, Ch 3, §3],

a) Fixed $i \in \{1, \dots, l\}$, a $\mathbb{Z}$-grading of $\mathcal{L}$ is defined making $\mathcal{L}_p = \oplus_{\alpha \in \Phi \cup \{0\}} \{L_\alpha \mid \alpha = \sum_{j=1}^{l} k_j \alpha_j,\ k_i = p\}$, so $\mathcal{L} = \mathcal{L}_{-n_i} \oplus \cdots \oplus \mathcal{L}_0 \oplus \cdots \oplus \mathcal{L}_{n_i}$ where $\mathcal{L}_0 = H \oplus \sum\{L_\alpha \mid \alpha = \sum_{j=1, j\neq i}^{l} k_j\alpha_j \in \Phi\}$ is reductive with one-dimensional center $Z = \{h \in H \mid \alpha_j(h) = 0 \quad \forall j \neq i\}$ and semisimple part with root system Φ_i and basis $\{\alpha_j \mid j \neq i\}$. Recall (2.2 a) that $\mathcal{L}_0$ is maximal if and only if $n_i = 1$, in which case we have a 3-grading $\mathcal{L} = \mathcal{L}_{-1} \oplus \mathcal{L}_0 \oplus \mathcal{L}_1$.

It is not true that the homogeneous components are irreducible in any $\mathbb{Z}$-grading. This one is determined by fixing integers $p_1, \dots, p_l \geq 0$, and defining $\mathcal{L}_p = \oplus_{\alpha \in \Phi \cup \{0\}} \{L_\alpha \mid \alpha = \sum_{j=1}^{l} k_j\alpha_j,\ \sum_{j=1}^{l} k_j p_j = p\}$. So the homogeneous component $\mathcal{L}_p$ decomposes into the direct sum of the irreducible modules $\mathcal{L}_p^{(\nu)}$ (see [OV, p. 108]), which are the sum of the root spaces L_α contained in $\mathcal{L}_p$ corresponding to the roots with fixed coefficients at the simple roots α_k such that $p_k \neq 0$. In our case, $p_i = 1$ and $p_j = 0$ if $j \neq i$, so $\mathcal{L}_p^{(\nu)}$ must have fixed coefficient at α_i, obviously p, and $\mathcal{L}_p^{(\nu)} = \mathcal{L}_p$ do be irreducible.

b) For $\alpha_0 \equiv -\tilde{\alpha}$ (the minimum root), any root $\alpha \in \Phi$ can be uniquely represented in the form $\alpha = \sum_{j=0}^{l} k_j\alpha_j$ for $0 \leq k_j \leq n_j$ (if $\alpha \in \Phi^+$, $k_0 = 0$, and if $\alpha \in \Phi^-$, $k_0 = 1$).

Now, for an index i with n_i a prime number ($\neq 1$), we define θ by means of $\theta|_{L_\alpha} = w^{k_i}\,\mathrm{id}$ if L_α is the root space associated to $\alpha = \sum_{j=0}^{l} k_j\alpha_j$ and $w = \exp\frac{2\pi I}{n_i} \in F$ is a primitive n_i-th root of the unity. Since θ is an inner automorphism of $\mathcal{L}$ of order n_i, we have a $\mathbb{Z}_{n_i}$-grading of $\mathcal{L}$ given by $\mathcal{L}_{\bar{p}} = \sum_{\alpha\in\Phi\cup\{0\}}\{L_\alpha \mid \alpha = \sum_{j=0}^{l} k_j\alpha_j,\ k_i \equiv p \,(\mathrm{mod}\, n_i)\}$, so that $\mathcal{L}_{\bar{0}}$ is the semisimple subalgebra of rank l with root system Γ_i and basis $\{\alpha_0, \alpha_j \mid j \neq i\}$, that is, the obtained one removing the node i of the extended Dynkin diagram. In fact, the condition of n_i to be prime is only necessary for the maximality of the subalgebra $\mathrm{Fix}\,\theta = \mathcal{L}_{\bar{0}}$.

These $\mathbb{Z}_m$-gradings are called of inner type (because they are produced by inner automorphisms of order m) and they are associated to the Kac diagrams Aff1, in contrast to the gradings produced by outer automorphisms, which are associated to the Kac diagrams Aff2 and Aff3. Again the components of a $\mathbb{Z}_m$-grading of inner type may be not irreducible. One such grading is given by integers $p_0, p_1, \ldots, p_l \geq 0$ such that $\sum_{j=0}^{l} n_jp_j = m$ ($n_0 = 1$), being the homogeneous components $\mathcal{L}_{\bar{p}} = \sum_{\alpha\in\Phi\cup\{0\}}\{L_\alpha \mid \alpha = \sum_{j=0}^{l} k_j\alpha_j,\ \sum_{j=0}^{l} k_jp_j \equiv p\,(\mathrm{mod}\, m)\}$ for $p \in \{0, \ldots, m-1\}$, direct sum of the irreducible components $\mathcal{L}_{\bar{p}}^{(\nu)}$ (see [OV, p. 113]). In our case $p_i = 1$ and $p_j = 0$ if $j \neq i$, so that there is only one $\mathcal{L}_{\bar{p}}^{(\nu)} \neq 0$ (the coefficient at α_i is $k_i = p$), and $\mathcal{L}_{\bar{p}} = \mathcal{L}_{\bar{p}}^{(\nu)}$ is irreducible, as we expected from Theorem 1.

In conclusion, we can determine the grading groups by looking at the labels of the removed nodes, taking care of the repeated factors.

3.5 *Example.*

Theorem 1 states that the maximal rank reductive subalgebras are associated to infinite grading groups, which seems to be contrary to the fact that the null component of any $\mathbb{Z}_n$-grading of inner type is just a maximal rank reductive subalgebra (according to [Ka] and 3.4.b, we choose $p_0, p_1, \ldots, p_l \geq 0$ such that $\sum_{j=0}^{l} n_jp_j = n$, so the Dynkin diagram of the semisimple part is obtained removing the nodes from the affine diagram Aff1 whose attached labels are nonzero, and the dimension of the center is the number of nonzero labels minus 1).

We next display a specific example to illustrate the situation. Taking as nonzero labels $p_0 = p_4 = 1$, it is produced a $\mathbb{Z}_3$-grading of F_4 whose fixed subalgebra is the direct sum of B_3 and a one-dimensional center Z. Let $\{h_i\}_{i=1}^4$ be a basis of a Cartan subalgebra of F_4 such that $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ given by $\alpha_1(h) = w_2 - w_3$, $\alpha_2(h) = w_3 - w_4$, $\alpha_3(h) = w_4$ and $\alpha_4(h) = \frac{1}{2}(-w_1 + w_2 + w_3 + w_4)$, for $h = \sum_{i=1}^4 w_ih_i$, is a basis of the root system Φ of F_4. Thus, $\Delta' = \{\alpha_1, \alpha_2, \alpha_3\}$ is a basis of the root system Φ' of B_3 (obviously contained in Φ), the center is $Z = Fh_1$, and the grading group of F_4 based on $\mathfrak{h} = B_3 \oplus Z$ is $G = \mathbb{Z}\Phi/\mathbb{Z}\Phi' = \{n\alpha_4 + \mathbb{Z}\Phi' \mid n \in \mathbb{Z}\} \approx \mathbb{Z}$. So, we obtain a $\mathbb{Z}$-grading of $\mathcal{L} = \mathcal{L}_{-2} \oplus \mathcal{L}_{-1} \oplus \mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2$, where each $\mathcal{L}_n \equiv \mathcal{L}_{n\alpha_4+\mathbb{Z}\Phi'} = \oplus\{L_\alpha \mid \alpha = \sum_{i=1}^4 k_i\alpha_i \in \Phi,\ k_4 = n\}$ is an $\mathfrak{h}$-irreducible module.

To obtain the decomposition of F_4 as a sum of $\mathfrak{h}$-irreducible modules, we have only to find the roots $\alpha \in \Phi$ such that $\alpha + \alpha_i \notin \Phi$ for $i = 1, 2, 3$, since the corresponding maximal vectors generate these modules (in fact, this is a general method to decompose a semisimple Lie algebra as a sum of $\mathfrak{h}$-irreducible modules for $\mathfrak{h}$ a regular subalgebra). There are five roots in such situation: $2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4$, $\alpha_1+2\alpha_2+3\alpha_3+\alpha_4$, $\alpha_1+2\alpha_2+2\alpha_3$ (which generates B_3), $-\alpha_4$ and $-\alpha_2-2\alpha_3-2\alpha_4$. For $h'=w_2h_2+w_3h_3+w_4h_4$ an arbitrary element in the Cartan subalgebra of B_3,

$$\begin{array}{l}(2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4)(h')=w_2=(\alpha_1+\alpha_2+\alpha_3)(h')=\lambda_1(h')\\ (-\alpha_2-2\alpha_3-2\alpha_4)(h')=w_2=\lambda_1(h')\\ (\alpha_1+2\alpha_2+3\alpha_3+\alpha_4)(h')=\frac{1}{2}(w_2+w_3+w_4)=\frac{1}{2}(\alpha_1+2\alpha_2+3\alpha_3)(h')=\lambda_3(h')\\ (-\alpha_4)(h')=\frac{1}{2}(w_2+w_3+w_4)=\lambda_3(h')\end{array}$$

and $z=-2h_1$ acts with eigenvalue n in $\mathcal{L}_n$; therefore the decomposition is

$$F_4=B_3\oplus Fz\oplus V(\lambda_1)^2\oplus V(\lambda_3)^1\oplus V(\lambda_3)^{-1}\oplus V(\lambda_1)^{-2}$$

where the super-index means the action of z, and $V(\lambda)$ denotes a basic module for B_3 (λ_i the fundamental weight).

The $\mathbb{Z}_3$-grading $F_4=M_{\bar 0}\oplus M_{\bar 1}\oplus M_{\bar 2}$ is obtained making $M_{\bar 2}=\mathcal{L}_2\oplus\mathcal{L}_{-1}$ and $M_{\bar 1}=\mathcal{L}_1\oplus\mathcal{L}_{-2}$, that is, taking the group epimorphism $\pi\colon\mathbb{Z}\to\mathbb{Z}_3$ and for any $g\in\mathbb{Z}_3$ the homogeneous component $M_g=\sum_{\pi(n)=g}\mathcal{L}_n$, as in Theorem 1.c.

4 Models of F_4 based on semisimple subalgebras of rank 4

We can now develop the models of F_4 based on the semisimple subalgebras of rank 4 which have not been mentioned in the introduction, namely, $2A_2$, $A_3\oplus A_1$, $C_2\oplus 2A_1$ and $4A_1$. These models will be valid over algebraically closed fields of characteristic zero, but they will also yield a split simple Lie algebra of dimension 52 for arbitrary fields of characteristic different from 2 and 3, hence, the split algebra F_4.

Taking into account the marked nodes that we have to remove (or simply looking at the decompositions of F_4 as sums of irreducible modules), we know the grading groups: the ones corresponding to $A_1\oplus C_3$, $2A_2$ and B_4 are $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_2$ respectively, and the one associated to $A_3\oplus A_1$ (obtained in only one step removing α_3 from $F_4^{(1)}$, although the label is 4, not prime) is $\mathbb{Z}_4$. From B_4 we obtain $2A_1\oplus C_2$ and D_4 (besides $A_1\oplus A_3$), both with grading groups $\mathbb{Z}_2^2$ (a priori a $\mathbb{Z}_2$-grading of another $\mathbb{Z}_2$-grading could lead to $\mathbb{Z}_2^2$ or $\mathbb{Z}_4$), and $2A_1\oplus C_2$ breaks in a new $\mathbb{Z}_2$-grading with even part $4A_1$, being $\mathbb{Z}_2^3$ the grading group related to it.

Consequently, in this section models of F_4 will be given where:

1. F_4 is $\mathbb{Z}_3$-graded with null component of type A_2.
2. F_4 is $\mathbb{Z}_4$-graded with null component of type $A_3\oplus A_1$.
3. F_4 is $\mathbb{Z}_2^2$-graded with null component of type $C_2\oplus 2A_1$.
4. F_4 is $\mathbb{Z}_2^3$-graded with null component of type $4A_1$.

4.1 A $\mathbb{Z}_3$-grading of F_4

Some $\mathbb{Z}_3$-gradings on exceptional Lie algebras are known. The first one arises from the $\mathbb{Z}_3$-grading on the split Cayley algebra $\mathcal{C}$ viewed as Zorn's vector matrix algebras (see [Ja3, p. 142]).

This fact provides the corresponding $\mathbb{Z}_3$-grading on the split Lie algebra of type G_2, the set of derivations of $\mathcal{C}$. In this way, for V a three-dimensional vector space,

$$\mathcal{L} = \mathrm{sl}(V) \oplus V \oplus V^*$$

is a Lie algebra of type G_2, where the set of trace zero endomorphisms $\mathrm{sl}(V)$ is a Lie subalgebra of type A_2, the actions of $\mathrm{sl}(V)$ on V and V^* are the natural ones (that is, V is the natural module and V^* its dual one), and

$$\begin{aligned}
[f,u] &= 3f(-)u - f(u)\,\mathrm{id}_V \\
[u,v] &= 2u \wedge v \\
[f,g] &= 2f \wedge g
\end{aligned} \tag{3}$$

for any $u, v \in V,\ f, g \in V^*$, where $u \wedge v$ denotes the element in V^* given by $\det(u, v, -)$ (fixed a nonzero trilinear alternating map $\det\colon V \times V \times V \to F$) and $f \wedge g$ denotes the element in V^* given by $\det^*(f, g, -)$, being $\det^*$ the dual map.

The other $\mathbb{Z}_3$-grading is given by a very nice and symmetric model of E_6 which appears in [A, chapter 13]. Adams uses three vector spaces V_1, V_2 and V_3 of dimension 3 (over $\mathbb{C}$) and their dual ones V_1^*, V_2^* and V_3^*, and thus

$$\mathcal{L} = \mathrm{sl}(V_1) \oplus \mathrm{sl}(V_2) \oplus \mathrm{sl}(V_3) \oplus\ V_1 \otimes V_2 \otimes V_3\ \oplus\ V_1^* \otimes V_2^* \otimes V_3^*$$

is a Lie algebra of type E_6, where $\sum \mathrm{sl}(V_i)$ is a Lie subalgebra of type $3A_2$, its actions on $V_1 \otimes V_2 \otimes V_3$ and $V_1^* \otimes V_2^* \otimes V_3^*$ are the natural ones (the ith simple ideal acts on the ith slot), and

$$\begin{aligned}
[\otimes f_i, \otimes u_i] &= \sum_{\substack{k=1,2,3 \\ i \neq j \neq k}} f_i(u_i) f_j(u_j)\big(f_k(-)u_k - \frac{1}{3} f(u_k)\,\mathrm{id}_{V_k}\big) \\
[\otimes u_i, \otimes v_i] &= \otimes(u_i \wedge v_i) \\
[\otimes f_i, \otimes g_i] &= \otimes(f_i \wedge g_i)
\end{aligned} \tag{4}$$

for any $u_i, v_i \in V_i,\ f_i, g_i \in V_i^*$, with the wedge products as in (3).

Both models work too for arbitrary fields of characteristic different than 2 and 3. Notice the analogy between them.

Now, in the *middle* of the previous models of G_2 and E_6, we find the following one of F_4, obtained as the set of fixed points of certain automorphism of E_6:

Theorem 2. *In the model* (4)*, let us take $\tau \in \operatorname{Aut} E_6$ induced by $\tau(v_1 \otimes v_2 \otimes v_3) = v_1 \otimes v_3 \otimes v_2$. If we take $V_1 = V_2 = V_3 = V = W$, and $S^2(W) =< \{v \cdot w + w \cdot v \mid v, w \in W\} >$ denotes the second symmetric power, then*

a) The Lie subalgebra of the fixed elements can be described as

$$\tilde{\mathcal{L}} = \mathrm{sl}(V) \oplus \mathrm{sl}(W) \oplus\ V \otimes S^2(W)\ \oplus\ V^* \otimes S^2(W^*) \tag{5}$$

where $\mathrm{sl}(V) \oplus \mathrm{sl}(W)$ is a Lie subalgebra, its actions on $V \otimes S^2(W)$ and $V^* \otimes S^2(W^*)$ are the natural ones, and

$$\begin{aligned}
[f \otimes h \cdot h, u \otimes w \cdot w] &= h(w)^2 \pi f_u + f(u) h(w) \pi h_w \\
[u \otimes w \cdot w, v \otimes x \cdot x] &= (u \wedge v) \otimes (w \wedge x) \cdot (w \wedge x) \\
[f \otimes h \cdot h, g \otimes j \cdot j] &= (f \wedge g) \otimes (h \wedge j) \cdot (h \wedge j)
\end{aligned}$$

for any $u, v \in V$, $w, x \in W$, $f, g \in V^*$, $h, j \in W^*$, *denoting by* $f_u \equiv f(-)u$ *and the projection* $\pi f \equiv f - \frac{1}{3}\operatorname{tr}(f)\,\mathrm{id}$.

b) If $\mathcal{L}_{\bar 0} = \mathrm{sl}(V) \oplus \mathrm{sl}(W)$, $\mathcal{L}_{\bar 1} = V \otimes S^2(W)$ *and* $\mathcal{L}_{\bar 2} = V^* \otimes S^2(W^*)$, *then* $\tilde{\mathcal{L}} = \sum_{i=0,1,2} \mathcal{L}_{\bar i}$ *is a* $\mathbb{Z}_3$*-grading.*

c) $\tilde{\mathcal{L}}$ *is a simple Lie algebra and* $\dim_F \tilde{\mathcal{L}} = 52$, *hence* $\tilde{\mathcal{L}}$ *is of type* F_4.

The proof of this theorem is immediate, once we have showed the simplicity of the resulting model (5). To check it, and be able to use it later in the other models, we will reproduce the general conditions of all the constructions in this work.

Lemma 1. *If the Lie algebra* $\mathcal{L} = \oplus_{g\in G}\mathcal{L}_g$ *is* G*-graded by a finite abelian group* G, *such that* $\mathfrak{h} \equiv \mathcal{L}_0$ *is a semisimple subalgebra,* $\{\mathcal{L}_g\}_{g\neq 0}$ *is a collection of pair-wise nonisomorphic* $\mathfrak{h}$*-irreducible modules and nonisomorphic to the simple ideals of* $\mathfrak{h}$, *verifying* $[\mathcal{L}_{g_1}, \mathcal{L}_{g_2}] \neq 0$ *for any* $g_1, g_2 \in G$ *such that* $g_1 + g_2 \neq 0$, *and* $\mathfrak{h} \subset [\mathfrak{m}, \mathfrak{m}]$ *for* $\mathfrak{m} \equiv \oplus_{g\in G\setminus\{0\}}\mathcal{L}_g$, *then* $\mathcal{L}$ *is simple.*

Proof. Let I be an ideal of $\mathcal{L}$. Since I is $\mathfrak{h}$-invariant ($\mathfrak{h} \subset \mathcal{L}$), there is $S \subset G \setminus \{0\}$ and $\tilde{\mathfrak{h}}$ an ideal of $\mathfrak{h}$ such that $I = \tilde{\mathfrak{h}} \oplus \sum_{s\in S}\mathcal{L}_s$. If there exists an element $s \in S$, for any $g \in G \setminus \{0\}$ the irreducibility implies that $\mathcal{L}_g = [\mathcal{L}_s, \mathcal{L}_{g-s}] \subset I$, hence $\mathfrak{m} \subset I$, $\mathfrak{h} \subset [\mathfrak{m}, \mathfrak{m}] \subset I$ and $I = \mathcal{L}$. Otherwise, S is empty, $I = \tilde{\mathfrak{h}}$ verifies $[I, \mathfrak{m}] = 0$, $[I, [\mathfrak{m}, \mathfrak{m}]] = 0$, $[I, \mathfrak{h}] = 0$ and $I \subset Z(\mathfrak{h}) = 0$. □

It is obvious that $\tilde{\mathcal{L}}$ in Theorem 1 satisfies the required conditions in the Lemma.

Remarks.

i) If in a vector space $\mathcal{L} = \mathfrak{h} \oplus \mathfrak{m}$ we define any bracket such that $\mathfrak{h}$ is a Lie subalgebra and $\mathfrak{m}$ is an $\mathfrak{h}$-module, directly we have $J(\mathfrak{h},\mathfrak{h},\mathfrak{h}) = 0 = J(\mathfrak{h},\mathfrak{h},\mathfrak{m})$, for $J(x, y, z) \equiv [[x, y], z] + [[y, z], x] + [[z, x], y]$. Besides the condition $J(\mathfrak{h},\mathfrak{m},\mathfrak{m}) = 0$ is equivalent to that the bracket $[\ ,\ ]|_{\mathfrak{m}\otimes\mathfrak{m}} \colon \mathfrak{m} \otimes \mathfrak{m} \to \mathcal{L}$ is a homomorphism of $\mathfrak{h}$-modules. Therefore, that bracket endows $\mathcal{L}$ with a Lie algebra structure if and only if $J(\mathfrak{m},\mathfrak{m},\mathfrak{m}) = 0$.

So, we can take the decomposition of F_4 as a sum of irreducible $\mathfrak{h}$-modules as a starting point to construct the Lie algebra F_4. Even more, it is not essential the preliminary knowledge of the decomposition of $\mathcal{L}$ as a direct sum of irreducible modules for its semisimple subalgebra $\mathfrak{h}$, any choice as in the preceding paragraph would yield a Lie algebra.

ii) Therefore, a way to approach to the above construction of F_4, alternative to Theorem 2, is to begin with the expression (5) (which appears, for example, in [LMa1, p. 25] or [LMa2, p. 80]), that is, the decomposition of F_4 as a sum of modules for its subalgebra of type $2A_2$. The uniqueness of the homomorphisms between the different components (Theorem 1.d) forces the existence of scalars $\alpha, \beta, \gamma, \delta$ such that $\tilde{\mathcal{L}} = \sum_{i=0,1,2} \mathcal{L}_{\bar i}$ is a Lie algebra with the bracket in $\mathfrak{m} = \mathcal{L}_{\bar 1} \oplus \mathcal{L}_{\bar 2}$ given by:

$$\begin{aligned}
[f\otimes h\cdot h, u\otimes w\cdot w] &= \alpha h(w)^2\pi f_u + \beta f(u)h(w)\pi h_w\\
[u\otimes w\cdot w, v\otimes x\cdot x] &= \gamma(u\wedge v)\otimes(w\wedge x)\cdot(w\wedge x)\\
[f\otimes h\cdot h, g\otimes j\cdot j] &= \delta(f\wedge g)\otimes(h\wedge j)\cdot(h\wedge j)
\end{aligned}$$

In order to find these scalars, we have only to impose $J(\mathfrak{m},\mathfrak{m},\mathfrak{m}) = 0$. Independently of them, $J(\mathcal{L}_1,\mathcal{L}_1,\mathcal{L}_1) = 0 = J(\mathcal{L}_2,\mathcal{L}_2,\mathcal{L}_2)$ holds. On the other hand,

$$\begin{aligned}
J(u\otimes x\cdot x, v\otimes w\cdot w, f\otimes g\cdot g) \;&= \\
= f(v)u\otimes\ & \big((\tfrac{1}{3}(\alpha+2\beta)-\gamma\delta)g(w)^2x\cdot x\\
&+2(-\beta+\gamma\delta)g(x)g(w)w\cdot x+(\alpha-\gamma\delta)g(x)^2w\cdot w\big)\\
-f(u)v\otimes\ & \big((\alpha-\gamma\delta)g(w)^2x\cdot x+2(-\beta+\gamma\delta)g(x)g(w)w\cdot x\\
&+(\tfrac{1}{3}(\alpha+2\beta)-\gamma\delta)g(x)^2w\cdot w\big)
\end{aligned}$$

is always equal to 0 if and only if $\alpha = \beta = \gamma\delta$. The identity $J(\mathcal{L}_2, \mathcal{L}_2, \mathcal{L}_1) = 0$ leads to the same conditions on the scalars. Therefore, taking $\alpha = \beta = \gamma = \delta = 1$ provides a Lie algebra structure in $\tilde{\mathcal{L}}$.

iii) There is a way of viewing $2A_2$ inside F_4 without using the Albert algebra. It is described in a symmetric version of the magic square [E, §4]: if S is a para-Hurwitz algebra of dimension 1, and S' is a Okubo algebra (a kind of 8-dimensional symmetric composition algebra), the algebra $\mathrm{tri}(S) \oplus \mathrm{tri}(S') \oplus (S \otimes S')_1 \oplus (S \otimes S')_2 \oplus (S \otimes S')_3$ is of type F_4 (based on D_4). If $\theta \in \mathrm{tri}\, S$, $\theta' \in \mathrm{tri}\, S'$ are the triality automorphisms, the map given by $\Psi|_{\mathrm{tri}\, S} = \theta$, $\Psi|_{\mathrm{tri}\, S'} = \theta'$, $\Psi((s\otimes s')_i) = (s\otimes s')_{i+1\,(\mathrm{mod}\,3)}$ defines an automorphism of order 3 in F_4, and its fixed subalgebra $\mathrm{Der}\, S' \oplus \iota(S \otimes S')$ (for $\iota(s \otimes s') = (s \otimes s')_1 + (s \otimes s')_2 + (s \otimes s')_3$) is of type $2A_2$.

4.2 A $\mathbb{Z}_4$-grading on F_4

Among the seven semisimple subalgebras of F_4 of rank 4, there are three ones which divide F_4 in 4 pieces, but there is only one that provides a $\mathbb{Z}_4$-grading, since not all the $\mathfrak{h}$-submodules of F_4 are selfdual: we are talking about $A_1 \oplus A_3$.

In order to describe the decomposition of $F_4 = \mathfrak{h} \oplus \mathfrak{m}$ for $\mathfrak{h} \approx A_1 \oplus A_3$, recall that each irreducible $\mathfrak{h}$-submodule of $\mathfrak{m}$ is the tensor product of an A_1-irreducible module by an A_3-irreducible module. But the decomposition of F_4 for its regular subalgebra of type A_3 [Dy, p. 199] is

$$F_4 = V(\lambda_1 + \lambda_3) \oplus 3V(0) \oplus 2V(\lambda_1) \oplus 3V(\lambda_2) \oplus 2V(\lambda_3),$$

where $V(\lambda_1)$, $V(\lambda_2)$ and $V(\lambda_3)$ are the basic modules for A_3, that is, the natural module W and its exterior powers $\bigwedge^2 W$ and $\bigwedge^3 W$, of dimensions 4, 6 and 4 respectively. If $V \otimes W$ is one of the summands of the decomposition, $V^* \otimes W^*$ is another one, since $\mathfrak{m}$ is a selfdual $\mathfrak{h}$-module. Therefore, there exist A_1-modules V and V' of dimensions 2 and 3 such that $\mathfrak{m} \approx V \otimes W \oplus V' \otimes \bigwedge^2 W \oplus V \otimes \bigwedge^3 W$. It is easy to check that V and V' do not contain trivial submodules, hence they are the A_1-modules of types $V(1)$ and $V(2)$ (where $V(n)$ denotes the only A_1-irreducible module of dimension $n + 1$).

Now, before giving the model related to this decomposition, let us describe the homomorphisms which will be used for the bracket. The ones among the A_3-modules are listed in the following straightforward Lemma:

Lemma 2. *If W is a four-dimensional vector space, and we identify the exterior power $\bigwedge^3 W$ with the* $\mathrm{sl}(W) \approx A_3$*-dual module W^* by means of the determinant (a fixed multilinear alternating map $\det\colon W^4 \to F$), $u \wedge v \wedge w \equiv \det(u, v, w, -)$, the following maps are homomorphisms of* $\mathrm{sl}(W)$*-modules:*

$$\begin{array}{ll}
W \otimes W \to \bigwedge^2 W, & u \otimes v \mapsto u \wedge v \\
W \otimes \bigwedge^2 W \to \bigwedge^3 W, & u \otimes v \wedge w \mapsto u \wedge v \wedge w \\
W \otimes \bigwedge^3 W \to \mathrm{End}_F W, & u \otimes f \mapsto f_u \equiv f(-)u \\
\mathrm{End}_F W \xrightarrow{\pi} sl_F(W), & G \mapsto \pi G \equiv G - \frac{1}{4}\,\mathrm{tr}\, G\, \mathrm{id}_W \quad (\mathrm{tr}\, f_u = f(u)) \\
W \otimes \bigwedge^3 W \to F, & u \otimes f \mapsto f(u) \\
\bigwedge^2 W \otimes \bigwedge^2 W \to \mathrm{End}_F W, & (u \wedge v) \otimes (w \wedge x) \mapsto (u \wedge v \wedge w)_x - (u \wedge v \wedge x)_w \\
\bigwedge^2 W \otimes \bigwedge^2 W \to F, & (u \wedge v) \otimes (w \wedge x) \mapsto \det(u, v, w, x) \\
\bigwedge^2 W \otimes \bigwedge^3 W \to W, & (u \wedge v) \otimes f \mapsto f(u)v - f(v)u \\
\bigwedge^3 W \otimes \bigwedge^3 W \to \bigwedge^2 W, & f \otimes (u \wedge v \wedge w) \mapsto f(u)v \wedge w + f(v)w \wedge u + f(w)u \wedge v \\
 & \qquad \equiv f \wedge (u \wedge v \wedge w)
\end{array}$$

for any $u, v, w, x \in W$, $f \in W^*$.

On the other hand, there is a practical model for A_1 and its modules of type $V(n)$, such that the homomorphisms are expressed in terms of transvections. We follow the notations of [D]. Let us denote by V_n the linear space over F of the homogeneous polynomials in the variables x and y of degree n. Identifying sl(2) with the subalgebra $L = \text{span}\langle\{x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}\}\rangle \leq \text{Der}(F[x, y])$, it acts naturally on V_n such that V_n is the irreducible sl(2)-module of dimension $n+1$, if the characteristic of F is 0.

For $f \in V_i$, $g \in V_j$, the transvection $(f, g)_q$ $(0 \leq q \leq i, j)$ is defined by

$$(f, g)_q = \frac{(i-q)!}{i!}\frac{(j-q)!}{j!}\sum_{k=0}^{q}\left((-1)^k\binom{q}{k}\frac{\partial^q f}{\partial x^{q-k}\,\partial y^k}\,\frac{\partial^q g}{\partial x^k\,\partial y^{q-k}}\right) \in V_{i+j-2q}\,.$$

Since the map $(\ ,\ )_q : V_i \times V_j \to V_{i+j-2q}$ is L-invariant, it induces a homomorphism of $\text{sl}(2, F)$-modules, hence so $(\ ,\ )_1 : V_2 \otimes V_m \to V_m$ is.

In particular we can identify $(V_2, (\ ,\ )_1)$ with the Lie algebra sl(2) and thus the action of sl(2) on its module V_1 is given by $\frac{1}{2}(\ ,\ )_1$. The other involved homomorphisms are then induced by $(\ ,\ )_0 : V_1 \times V_1 \to V_2$ (the product of the polynomials), $(\ ,\ )_1 : V_1 \times V_1 \to F$ and $(\ ,\ )_2 : V_2 \times V_2 \to F$.

Theorem 3. *Under the notations above, if*

$$\begin{aligned}
\mathcal{L}_0 = &\quad V_2 \oplus \text{sl}(W)\\
\mathcal{L}_1 = &\quad V_1 \otimes W\\
\mathcal{L}_2 = &\quad V_2 \otimes \textstyle\bigwedge^2 W\\
\mathcal{L}_3 = &\quad V_1 \otimes \textstyle\bigwedge^3 W
\end{aligned}$$

and we define in $\mathcal{L} = \sum_{i=0}^{3} \mathcal{L}_i$ an anticommutative product given by the natural action of $\mathcal{L}_0$ on $\mathcal{L}_i$ $(i = 0, 1, 2, 3)$ and

$$\begin{aligned}
&[a \otimes v, b \otimes w] &&= \alpha ab \otimes v \wedge w\\
&[a \otimes v, b \otimes f] &&= \gamma f(v)ab + \delta(a, b)_1 \pi f_v\\
&[a \otimes f, b \otimes g] &&= \beta ab \otimes f \wedge g\\
&[a \otimes v, A \otimes (w \wedge x)] &&= \mu(a, A)_1 \otimes v \wedge w \wedge x\\
&[a \otimes f, A \otimes (w \wedge x)] &&= \eta(a, A)_1 \otimes (f(w)x - f(x)w)\\
&[A \otimes u \wedge v, B \otimes w \wedge x] &&= \theta \det(u, v, w, x)(A, B)_1 + \varphi(A, B)_2 \pi\big((u \wedge v \wedge w)_x - (u \wedge v \wedge x)_w\big)
\end{aligned}$$

for fixed scalars $\{\alpha, \beta, \gamma, \delta, \mu, \eta, \theta, \varphi\} \subset F \setminus \{0\}$ and for any $a, b \in V_1$, $A, B \in V_2$, $u, v, w, x \in W$ and $f, g \in W^*$, the obtained algebra verifies the Jacobi identity if and only if $\alpha\eta = \gamma = -\delta = -\beta\mu$ and $\theta = 2\varphi = -2\eta\mu$.

In particular, taking $\alpha = \beta = \delta = \mu = \varphi = 1$, $\gamma = \eta = -1$ and $\theta = 2$, $\mathcal{L}$ is a simple Lie algebra of dimension 52, of type F_4.

Proof. We have only to check $J(\mathcal{L}_i, \mathcal{L}_j, \mathcal{L}_k) = 0$ for any $i, j, k \in \{1, 2, 3\}$ because of Lemma 1 about the simplicity.

Let us denote $a, b, c \in V_1$, $A, B, C \in V_2$, $u, v, w, x, y, z \in W$ and $f, g, h \in W^*$. We will use a family of identities relative to homogeneous polynomials and transvections, called Gordan identities [D, p. 111]. For $F \in V_m$, $G \in V_n$, $H \in V_p$, the identity:

$$\sum_{i\geq 0} \frac{\binom{n-\alpha_1-\alpha_3}{i}\binom{\alpha_2}{i}}{\binom{m+n-2\alpha_3-i+1}{i}} \big((F,G)_{\alpha_3+i},H\big)_{\alpha_1+\alpha_2-i} = (-1)^{\alpha_1} \sum_{i\geq 0} \frac{\binom{p-\alpha_1-\alpha_2}{i}\binom{\alpha_3}{i}}{\binom{m+p-2\alpha_2-i+1}{i}} \big((F,H)_{\alpha_2+i},G\big)_{\alpha_1+\alpha_3-i}$$

will be denoted by $\begin{pmatrix} F & G & H \\ m & n & p \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix}$. For example, $\begin{pmatrix} a & b & c \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ means that

$$(ab,c)_1 + \frac{1}{2}(a,b)_1 c = (a,c)_1 b, \tag{6}$$

and permuting in this identity a and b, and then adding and subtracting both expressions, we get

$$2(ab,c)_1 = (a,c)_1 b + (b,c)_1 a \tag{7}$$

$$(a,b)_1 c + (b,c)_1 a + (c,a)_1 b = 0, \tag{8}$$

since $(\ ,\ )_1$ is skew. Besides adding (6) cyclicly in a, b, c, we obtain

$$(ab,c)_1 + (bc,a)_1 + (ca,b)_1 = 0, \tag{9}$$

so that,

$$J(a\otimes u, b\otimes v, c\otimes w) = -\alpha\mu\big((c,ab)_1 + (a,bc)_1 + (b,ca)_1\big)\otimes u\wedge v\wedge w = 0$$

and $J(\mathcal{L}_1,\mathcal{L}_1,\mathcal{L}_1) = 0$ without restrictions on the scalars.

If we consider another case, $(i,j,k) = (1,1,3)$,

$$\begin{aligned} J(a\otimes v, b\otimes w, c\otimes f) \ &= \big(\alpha\eta(ab,c)_1 + \delta(b,c)_1 a - \tfrac{1}{2}\gamma(ac,b)_1 + \tfrac{1}{4}\delta(a,c)_1 b\big)\otimes f(v)w \\ &+\big(-\alpha\eta(ab,c)_1 + \tfrac{1}{2}\gamma(bc,a)_1 - \tfrac{1}{4}\delta(b,c)_1 a - \delta(a,c)_1 b\big)\otimes f(w)v \end{aligned}$$

will be 0 if the parentheses are. By (7) and (8), the first parenthesis is

$$(b,c)_1 a\left(\frac{1}{2}\alpha\eta + \delta + \frac{1}{2}\gamma\right) + (a,c)_1 b\left(\frac{1}{2}\alpha\eta - \frac{1}{4}\gamma + \frac{1}{4}\delta\right),$$

which is 0 if and only if

$$\alpha\eta = \gamma = -\delta. \tag{10}$$

The same conditions are obtained making null the second parenthesis.

Now, for $(i,j,k) = (1,1,2)$,

$$\begin{aligned} &J(a\otimes v, b\otimes w, A\otimes x\wedge y) \\ &\quad = \big(\theta\alpha\det(v,w,x,y)(ab,A)_1 - \mu\gamma\det(w,x,y,v)a(b,A)_1 + \mu\gamma\det(v,x,y,w)b(a,A)_1\big) \\ &\qquad + \big(\varphi\alpha(ab,A)_2\pi((v\wedge w\wedge x)_y - (v\wedge w\wedge y)_x) \\ &\qquad\qquad -\mu\delta(a,(b,A)_1)_1\pi((w\wedge x\wedge y)_v) + \mu\delta(b,(a,A)_1)_1\pi((v\wedge x\wedge y)_w)\big) \end{aligned}$$

will be zero if its projections on $\mathrm{sl}(V)$ and $\mathrm{sl}(W)$ are zero.

Using the Gordan identity $\begin{pmatrix} a & A & b \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, we obtain

$$(a, A)_1 b = (ab, A)_1 + \frac{1}{2}(a, b)_1 A. \tag{11}$$

Again permuting a and b, and afterwards adding and subtracting,

$$(a, A)_1 b + (b, A)_1 a = 2(ab, A)_1 \tag{12}$$
$$(a, A)_1 b - (b, A)_1 a = (a, b)_1 A. \tag{13}$$

Hence, the projection on $\mathrm{sl}(V)$ is $\det(v, w, x, y)\left(\frac{1}{2}\theta\alpha + \gamma\mu\right)\big((a, A)_1 b + (b, A)_1 a\big)$, which is null if and only if

$$\theta\alpha = -2\gamma\mu \tag{14}$$

Now the identity $\begin{pmatrix} a & A & b \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}$, that is,

$$(ab, A)_2 = (a, (b, A)_1)_1 = (b, (a, A)_1)_1, \tag{15}$$

implies that the projection on $\mathrm{sl}(W)$ is

$$(ab, A)_2\big(\varphi\alpha\pi((v \wedge w \wedge x)_y - (v \wedge w \wedge y)_x) - \mu\delta\pi((w \wedge x \wedge y)_v - (v \wedge x \wedge y)_w)\big),$$

but

$$(v \wedge w \wedge x)_y - (w \wedge x \wedge y)_v + (x \wedge y \wedge v)_w - (y \wedge v \wedge w)_x = \det(v, w, x, y)\,\mathrm{id}_W, \tag{16}$$

hence $\pi((v \wedge w \wedge x)_y - (v \wedge w \wedge y)_x) = \pi((w \wedge x \wedge y)_v - (v \wedge x \wedge y)_w)$ and the desired condition is equivalent to

$$\varphi\alpha = \mu\delta. \tag{17}$$

Notice that (14) and (17) are equivalent to $\theta = -2\mu\eta = 2\varphi$, supposed (10).

For $(i, j, k) = (1, 2, 3)$, we will use again (11) y (13). Thus

$$\begin{aligned} &J(a \otimes u, A \otimes w \wedge x, b \otimes f) \\ &\quad = \big((a, A)_1 b(-\beta\mu - \gamma) + (a, b)_1 A\left(\tfrac{1}{2}\gamma + \tfrac{1}{2}\delta\right)\big) \otimes f(u) w \wedge x \\ &\quad + \big((a, A)_1 b(-\beta\mu + \delta) + (b, A)_1 a(-\eta\alpha - \delta)\big) \otimes f(w) x \wedge u \\ &\quad + \big((a, A)_1 b(-\beta\mu + \delta) + (b, A)_1 a(-\eta\alpha - \delta)\big) \otimes f(x) u \wedge w \end{aligned}$$

is zero if and only if $\beta\mu = \delta = -\eta\alpha = -\gamma$, that is, the only new condition is

$$\beta\mu = \delta.$$

We have already showed the necessity of the conditions about the scalars in the statement of the theorem, but we have still to check the remaining identities $J(\mathcal{L}_i, \mathcal{L}_j, \mathcal{L}_k) = 0$ for the sufficiency.

As in case $i = j = k = 1$, the case $i = j = k = 3$ is true independently of the scalars, because of (9), and so as $i = j = k = 2$, using the identities

$$\det(w,y,z,x)u\wedge v+\det(w,y,z,v)x\wedge u+\det(w,y,z,u)v\wedge x$$
$$+\det(x,u,v,w)y\wedge z+\det(x,u,v,y)z\wedge w+\det(x,u,v,z)w\wedge y=0$$

and $((B,C)_1,A)_1=\frac{1}{2}((B,A)_2C-(C,A)_2B)$, obtained from $\begin{pmatrix} A & B & C\\ 2 & 2 & 2\\ 0 & 1 & 1\end{pmatrix}$ and from the Jacobi identity in $(V_2,(\ ,\ )_1)$.

On the other hand, the identity $\begin{pmatrix} A & a & B\\ 2 & 1 & 2\\ 0 & 1 & 1\end{pmatrix}$ gives

$$((A,a)_1,B)_1=((A,B)_1,a)_1+\frac{1}{2}((A,B)_2,a)_0,$$

and now permuting a and b, adding and subtracting, we have

$$((A,a)_1,B)_1+((B,a)_1,A)_1=(A,B)_2a \tag{18}$$
$$((A,a)_1,B)_1-((B,a)_1,A)_1=2((A,B)_1,a)_1. \tag{19}$$

Using them,

$$\begin{aligned}
&J(a\otimes u,A\otimes v\wedge w,B\otimes x\wedge y)\\
&\quad=\mu\eta((a,A)_1,B)_1\otimes(\det(u,v,w,x)y-\det(u,v,w,y)x)\\
&\qquad+\theta\det(v,w,x,y)\tfrac{1}{2}\tfrac{1}{2}\big(((A,a)_1,B)_1-((B,a)_1,A)_1\big)\otimes u\\
&\qquad+\varphi\big(((A,a)_1,B)_1+((B,a)_1,A)_1\big)\otimes(\pi((v\wedge w\wedge x)_y-(v\wedge w\wedge y)_x)(u))\\
&\qquad-\mu\eta((a,B)_1,A)_1\otimes(\det(u,x,y,v)w-\det(u,x,y,w)v)\\
&\quad=((A,a)_1,B)_1\otimes\big(-\mu\eta\det(u,v,w,x)y+\mu\eta\det(u,v,w,y)x+\tfrac{1}{4}\theta\det(v,w,x,y)u\\
&\qquad\qquad+\varphi\pi((v\wedge w\wedge x)_y-(v\wedge w\wedge y)_x)(u)\big)\\
&\qquad+((B,a)_1,A)_1\otimes\big(-\tfrac{1}{4}\theta\det(v,w,x,y)u+\mu\eta\det(u,x,y,v)w-\mu\eta\det(u,x,y,w)v\\
&\qquad\qquad+\varphi\pi((v\wedge w\wedge x)_y-(v\wedge w\wedge y)_x)(u)\big)
\end{aligned}$$

but

$$\begin{aligned}
&\det(v,w,x,u)y-\det(v,w,y,u)x-\tfrac{1}{2}\det(v,w,x,y)u=\pi((v\wedge w\wedge x)_y-(v\wedge w\wedge y)_x)(u)\\
&=-\pi((x\wedge y\wedge v)_w-(x\wedge y\wedge w)_v)(u),
\end{aligned}$$

so the above expression is 0 because $-\mu\eta-\varphi=0=\frac{1}{4}\theta-\frac{1}{2}\varphi$, and $J(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_2)=0$.

For the following case, $(i,j,k)=(1,3,3)$, note that

$$v\wedge(f\wedge g)=f(v)g-g(v)f,$$

(notation $f\wedge g$ as in Lemma 2), so,

$$\begin{aligned}
&J(a\otimes v,b\otimes f,c\otimes g)\\
&\quad=\big(\tfrac{1}{2}\gamma(ab,c)_1+\tfrac{1}{4}\delta(a,b)_1c-\mu\beta(a,bc)_1+\delta(a,c)_1b\big)\otimes f(v)g\\
&\quad+\big(-\delta(a,b)_1c-\mu\beta(a,bc)_1-\tfrac{1}{2}\gamma(ac,b)_1-\tfrac{1}{4}\delta(a,c)_1b\big)\otimes g(v)f,
\end{aligned}$$

which is equal to 0 because of (6) and $-\gamma=\delta=\mu\beta$.

Denoting $u \wedge v \wedge w \wedge x \equiv \det(u, v, w, x)$, it is easy to see that

$$f \wedge g \wedge x \wedge y = f(x)g(y) - f(y)g(x)$$

and so, using the restrictions on the scalars, the projection of $J(a \otimes f, b \otimes g, A \otimes x \wedge y) \in J(\mathcal{L}_3, \mathcal{L}_3, \mathcal{L}_2)$ on $\mathrm{sl}(V)$ is

$$\eta\gamma(f(x)g(y) - f(y)g(x))(2(ab, A)_1 - (b, A)_1 a - (a, A)_1 b) = 0$$

by (12), and its projection on $\mathrm{sl}(W)$, by (15), is

$$\eta\gamma(ab, A)_2 \pi(g_{f(x)y} - g_{f(y)x} - f_{g(x)y} + f_{g(y)x} + f_{g(x)y-g(y)x} - g_{f(x)y-f(y)x}) = 0.$$

Finally, for $(i, j, k) = (3, 2, 2)$, by (18) and (19),

$$\begin{aligned} &J(a \otimes f, A \otimes v \wedge w, B \otimes x \wedge y) \\ &\quad = \varphi((A, a)_1, B)_1 \otimes \big(f(v)w \wedge x \wedge y - f(w)v \wedge x \wedge y + \tfrac{1}{2}\det(v, w, x, y)f \\ &\qquad\qquad - f \circ \pi((v \wedge w \wedge x)_y - (v \wedge w \wedge y)_x)\big) \\ &\quad\ - \varphi((B, a)_1, A)_1 \otimes \big(f(x)y \wedge v \wedge w - f(y)x \wedge v \wedge w + \tfrac{1}{2}\det(v, w, x, y)f \\ &\qquad\qquad + f \circ \pi((v \wedge w \wedge x)_y - (v \wedge w \wedge y)_x)\big) \end{aligned}$$

will be 0 since $f(v)w \wedge x \wedge y = f \circ (w \wedge x \wedge y)_v$ and (16). □

4.3 A $\mathbb{Z}_2^2$-grading on F_4

The model of F_4 based on D_4 can be viewed through the magic square [E, §3], whereas the one based on $C_2 \oplus 2A_1$, the other grading over the Klein group, provides a different perspective.

There is only one subalgebra of type B_2 in F_4, which is regular and gives the decomposition

$$F_4 = V(\lambda_2) \oplus 6V(0) \ \oplus\ 4V(\lambda_1) \oplus 4V(\lambda_2);$$

hence the B_2-modules involved in our decomposition are only the basic ones: the natural and the spin modules. For an easier point of view, let us take V a four-dimensional vector space, $\mathrm{sp}(V, \varphi) \equiv \mathrm{sp}(V)$ the symplectic algebra with respect to a skewsymmetric bilinear form φ. It is of type $C_2 \approx B_2$, and the above models through this isomorphism are the natural one, V, and $\mathrm{sym}_0\, V = \{h \in \mathrm{End}_F\, V \mid \mathrm{tr}(h) = 0,\ \varphi(h(x), y) = \varphi(x, h(y))\ \forall x, y \in V\}$. Hence the brackets in our model will be expressed in terms of the homomorphisms between these modules, which are

Lemma 3. *The following maps are homomorphisms of* $\mathrm{sp}(V)$*-modules*

$$\begin{array}{ll} \mathrm{sp}(V) \times V \to V & (f, v) \mapsto f(v) \\ \mathrm{sp}(V) \times \mathrm{sym}_0\, V \to \mathrm{sym}_0\, V & (f, g) \mapsto [f, g] \\ V \times V \to \mathrm{sym}\, V & (u, v) \mapsto \sigma_{u,v} \equiv \varphi(u, -)v - \varphi(v, -)u \\ V \times V \to \mathrm{sp}(V) & (u, v) \mapsto \varphi_{u,v} \equiv \varphi(u, -)v + \varphi(v, -)u \\ V \times V \to F & (u, v) \mapsto \varphi(u, v) \\ \mathrm{sym}_0\, V \times V \to V & (f, u) \mapsto f(u) \\ \mathrm{sym}_0\, V \times \mathrm{sym}_0\, V \to \mathrm{sp}(V) & (f, g) \mapsto [f, g] \\ \mathrm{sym}_0\, V \times \mathrm{sym}_0\, V \to F & (f, g) \mapsto \mathrm{tr}(fg) \end{array}$$

Proposition 2. *Let V be a four-dimensional vector space, and A and B be two-dimensional vector spaces. Let $\varphi_V\colon V\times V\to F$, $\varphi_A\colon A\times A\to F$ and $\varphi_B\colon B\times B\to F$ be skewsymmetric bilinear forms, all of them denoted by φ. We define a $\mathbb{Z}_2^2$-graded anticommutative algebra $\mathcal{L}=\sum_{(i,j)\in\mathbb{Z}_2^2}\mathcal{L}_{(i,j)}$ such that*

$$\begin{aligned}\mathcal{L}_{(0,0)}&=\mathrm{sp}(A)\oplus\mathrm{sp}(V)\oplus\mathrm{sp}(B)\\ \mathcal{L}_{(1,0)}&=A\otimes V\\ \mathcal{L}_{(0,1)}&=V\otimes B\\ \mathcal{L}_{(1,1)}&=A\otimes\mathrm{sym}_0 V\otimes B\end{aligned}$$

where $\mathcal{L}_{(0,0)}$ is a Lie subalgebra of $\mathcal{L}$, the brackets $[\mathcal{L}_{(0,0)},\mathcal{L}_{(i,j)}]$ are given by the natural action as modules, and

$$\begin{aligned}
&[a\otimes u,a'\otimes v] &&=\varepsilon_1\varphi(a,a')\varphi_{u,v}+\varepsilon_2\varphi(u,v)\varphi_{a,a'}\\
&[a\otimes u,a'\otimes f\otimes b] &&=\gamma_1\varphi(a,a')f(u)\otimes b\\
&[a\otimes u,v\otimes b] &&=\gamma_2 a\otimes\pi\sigma_{u,v}\otimes b\\
&[u\otimes b,a\otimes f\otimes b'] &&=\gamma_3\varphi(b,b')a\otimes f(u)\\
&[u\otimes b,v\otimes b'] &&=\varepsilon_3\varphi(b,b')\varphi_{u,v}+\varepsilon_4\varphi(u,v)\varphi_{b,b'}\\
&[a\otimes f\otimes b,a'\otimes g\otimes b'] &&=\varepsilon_5\operatorname{tr}(fg)\varphi(a,a')\varphi_{b,b'}+\varepsilon_6\operatorname{tr}(fg)\varphi(b,b')\varphi_{a,a'}+\varepsilon_7\varphi(a,a')\varphi(b,b')[f,g]
\end{aligned}$$

for fixed scalars $\{\varepsilon_i\mid i=1,\dots,7\}\cup\{\gamma_i\mid i=1,2,3\}\subset F\setminus\{0\}$ and for any $a,a'\in A$, $b,b'\in B$, $u,v\in V$, $f,g\in\mathrm{sym}_0 V$, with $\pi\colon\operatorname{End}V=F\,\mathrm{id}\oplus\mathrm{sl}(V)\to\mathrm{sl}(V)$ the projection $\pi(f)=f-\frac14\operatorname{tr}(f)\,\mathrm{id}_V$.

Then $\mathcal{L}$ is a Lie algebra if and only if $\varepsilon_1=\varepsilon_2$, $\varepsilon_3=\varepsilon_4$, $\varepsilon_5=\varepsilon_6=-\frac12\varepsilon_7$, $\gamma_1\gamma_3=4\varepsilon_5$, $\gamma_1\gamma_2=2\varepsilon_1$, $\gamma_2\gamma_3=2\varepsilon_3$.

Proof. We have to check the Jacobi identity $J(\mathcal{L}_i,\mathcal{L}_j,\mathcal{L}_k)=0$ for any $i,j,k\in\{1,2,3\}$, where we denote $\mathcal{L}_1\equiv\mathcal{L}_{(1,0)}$, $\mathcal{L}_2\equiv\mathcal{L}_{(1,1)}$ and $\mathcal{L}_3\equiv\mathcal{L}_{(0,1)}$ for short. Besides we denote $\tilde\sigma_{v,w}\equiv\pi\sigma_{v,w}$ for any $v,w\in V$.

Case 1,1,3):

$J(a\otimes u,a'\otimes v,w\otimes b)=\varphi(a,a')\big(\varepsilon_1\varphi_{u,v}(w)+\gamma_2\gamma_1(-\tilde\sigma_{v,w}(u)+\tilde\sigma_{w,u}(v))\big)\otimes b$, but $-\tilde\sigma_{v,w}(u)+\tilde\sigma_{w,u}(v)=-\frac12\varphi_{u,v}(w)$, so that the condition is equivalent to

$$\gamma_1\gamma_2=2\varepsilon_1,$$

and analogously for case 3,3,1),

$$\gamma_3\gamma_2=2\varepsilon_3.$$

Case 1,1,1):

$$J(a_1\otimes u_1,a_2\otimes u_2,a_3\otimes u_3)=\sum_{\substack{\text{cyclic}\\1,2,3}}\big(\varepsilon_1\varphi(a_1,a_2)a_3\otimes\varphi_{u_1,u_2}(u_3)+\varepsilon_2\varphi_{a_1,a_2}(a_3)\otimes\varphi(u_1,u_2)u_3\big),$$

and, since $\sum_{\text{cyclic }1,2,3}\varphi(a_1,a_2)a_3=0$ (A is two-dimensional), we have

$$\sum_{\substack{\text{cyclic}\\1,2,3}}\big(\varphi(a_1,a_2)a_3\otimes\varphi_{u_1,u_2}(u_3)+\varphi_{a_1,a_2}(a_3)\otimes\varphi(u_1,u_2)u_3\big)=0,$$

and the identity is satisfied if and only if $\varepsilon_1=\varepsilon_2$, and similarly for case 3,3,3), if $\varepsilon_3=\varepsilon_4$.

Case 1,1,2): Under the conditions found until now,

$$\begin{aligned} J(a_1\otimes u, a_2\otimes v, a_3\otimes f\otimes b) &= \varepsilon_1\big(\varphi(a_1,a_2)(a_3)\otimes[\varphi_{u,v},f]+\varphi_{a_1,a_2}(a_3)\otimes\varphi(u,v)f\\ &\qquad +2\varphi(a_2,a_3)a_1\otimes\tilde\sigma_{f(v),u}+2\varphi(a_3,a_1)a_2\otimes\tilde\sigma_{f(u),v}\big)\otimes b\\ &= \varepsilon_1\big(\varphi(a_2,a_3)a_1\otimes(2\tilde\sigma_{f(v),u}-[\varphi_{u,v},f]+\varphi(u,v)f)\\ &\qquad +\varphi(a_3,a_1)a_2\otimes(2\tilde\sigma_{f(u),v}-[\varphi_{u,v},f]-\varphi(u,v)f)\big)\otimes b. \end{aligned}$$

Notice that we have the following identity

$$\sum_{\text{cyclic } w,x,u}(\varphi(v,w)\sigma_{x,u}+\varphi(w,u)\sigma_{x,v})=(\varphi(v,w)\varphi(x,u)+\varphi(v,x)\varphi(u,w)+\varphi(v,u)\varphi(w,x))\,\mathrm{id}_V,$$

as is easy to check evaluating in v and w (x and u play the same role than w) and using that $\dim V=4$. Hence,

$$\begin{aligned} \tilde\sigma_{\tilde\sigma_{w,x}(v),u}-\tilde\sigma_{\tilde\sigma_{w,x}(u),v} &= \varphi(w,v)\sigma_{x,u}-\varphi(x,v)\sigma_{w,u}-\tfrac12\varphi(w,x)\sigma_{v,u}-\varphi(w,u)\sigma_{x,v}\\ &\quad +\varphi(x,u)\sigma_{w,v}+\tfrac12\varphi(w,x)\sigma_{u,v}-\tfrac12\varphi(\tilde\sigma_{w,x}(v),u)\,\mathrm{id}+\tfrac12\varphi(\tilde\sigma_{w,x}(u),v)\,\mathrm{id}\\ &= -\varphi(u,v)\sigma_{w,x}-(\varphi(v,w)\varphi(x,u)+\varphi(v,x)\varphi(u,w)+\varphi(v,u)\varphi(w,x))\,\mathrm{id}\\ &\quad -\tfrac12(\varphi(w,v)\varphi(x,u)-\varphi(x,v)\varphi(w,u)-\tfrac12\varphi(w,x)\varphi(v,u))\,\mathrm{id}\\ &\quad +\tfrac12(\varphi(w,u)\varphi(x,v)-\varphi(x,u)\varphi(w,v)-\tfrac12\varphi(w,x)\varphi(u,v))\,\mathrm{id}\\ &= -\varphi(u,v)\sigma_{w,x}+\tfrac12\varphi(u,v)\varphi(w,x)\,\mathrm{id}=-\varphi(u,v)\tilde\sigma_{w,x}, \end{aligned}$$

but the maps $\tilde\sigma_{w,x}$ generate the vector space $\mathrm{sym}_0 V$, so we have checked

$$\tilde\sigma_{f(u),v}-\tilde\sigma_{f(v),u}=\varphi(u,v)f \tag{20}$$

On the other hand, $\tilde\sigma_{f(u),v}+\tilde\sigma_{f(v),u}=\sigma_{f(u),v}+\sigma_{f(v),u}=[\varphi_{u,v},f]$ (it is direct since $(\varphi(v,-)u)\circ f=\varphi(f(v),-)u$ and $f\circ(\varphi(v,-)u)=\varphi(v,-)f(u)$), so adding it to (20) we obtain $2\tilde\sigma_{f(u),v}=\varphi(u,v)f+[\varphi_{u,v},f]$ and permuting u and v we finish this case. It works too in the case 3,3,2).

Case 1,2,3):

$$\begin{aligned} J(a\otimes u, a'\otimes f\otimes b', v\otimes b) &= \varphi(b,b')\varphi_{a,a'}\big(\gamma_2\varepsilon_6\,\mathrm{tr}(\tilde\sigma_{v,u}\circ f)-\gamma_3\varepsilon_2\varphi(f(v),u)\big)\\ &\quad +\varphi(a,a')\varphi_{b,b'}\big(\gamma_2\varepsilon_5\,\mathrm{tr}(\tilde\sigma_{v,u}\circ f)+\gamma_1\varepsilon_4\varphi(f(u),v)\big)\\ &\quad +\varphi(a,a')\varphi(b,b')\big(\gamma_2\varepsilon_7[\tilde\sigma_{v,u},f]-\gamma_1\varepsilon_3\varphi_{f(u),v}+\gamma_3\varepsilon_1\varphi_{f(v),u}\big), \end{aligned}$$

but $[\tilde\sigma_{v,u},f]=\varphi_{f(v),u}-\varphi_{f(u),v}$ and $\mathrm{tr}(\tilde\sigma_{v,u}\circ f)=2\varphi(f(v),u)=-2\varphi(f(u),v)$, so that the Jacobi identity is equivalent to

$$\begin{aligned} \gamma_3\varepsilon_1&=\gamma_1\varepsilon_3=-\gamma_2\varepsilon_7\\ \gamma_3\varepsilon_2&=2\gamma_2\varepsilon_6\\ \gamma_1\varepsilon_4&=2\gamma_2\varepsilon_5, \end{aligned}$$

which, assumed the previously obtained restrictions on the scalars, are equivalent to

$$\gamma_1\gamma_3=4\varepsilon_5,\qquad \varepsilon_5=\varepsilon_6=-\frac12\varepsilon_7.$$

Case 1,2,2):

$$\begin{aligned} J(a\otimes f\otimes b,\ & a'\otimes g\otimes b', c\otimes u)\\ &= \varepsilon_6\,\mathrm{tr}(fg)\varphi(b,b')\varphi_{a,a'}(c)\otimes u+\varepsilon_7\varphi(a,a')\varphi(b,b')c\otimes[f,g](u)\\ &\quad +\gamma_1\gamma_3\varphi(a',c)\varphi(b',b)a\otimes f(g(u))+\gamma_1\gamma_3\varphi(c,a)\varphi(b,b')a'\otimes g(f(u))\\ &= \varphi(b,b')\varepsilon_6\big(\varphi(a,c)a'\otimes(-2[f,g](u)-4g(f(u))+\mathrm{tr}(fg)u)\\ &\qquad +\varphi(a',c)a\otimes(2[f,g](u)-4f(g(u))+\mathrm{tr}(fg)u)\big) \end{aligned}$$

Notice that $\tilde\sigma_{u,v}\circ g+g\circ\tilde\sigma_{u,v}=\sigma_{g(u),v}+\sigma_{u,g(v)}-\varphi(u,v)g\in F\,\mathrm{id}_V$ (by (20), its projection is $\tilde\sigma_{g(u),v}-\tilde\sigma_{g(v),u}-\varphi(u,v)g=0$), that is, $f\circ g+g\circ f=\frac14\operatorname{tr}(f\circ g+g\circ f)\,\mathrm{id}_V=\frac12\operatorname{tr}(f\circ g)\,\mathrm{id}_V$, hence $2f\circ g=f\circ g+g\circ f+[f,g]=[f,g]+\frac12\operatorname{tr}(fg)\,\mathrm{id}_V$, so that $2[f,g](u)-4fg(u)+\operatorname{tr}(fg)u=0=-2[f,g](u)-4gf(u)+\operatorname{tr}(fg)u$.

Finally, the case 2,2,2):

$$J(a_1\otimes f_1\otimes b_1,a_2\otimes f_2\otimes b_2,a_3\otimes f_3\otimes b_3)=\varepsilon_5\sum\nolimits_{\text{cyclic}}\big(\varphi(a_1,a_2)a_3\otimes\operatorname{tr}(f_1f_2)f_3\otimes\varphi_{b_1,b_2}(b_3)$$
$$+\varphi_{a_1,a_2}(a_3)\otimes\operatorname{tr}(f_1f_2)f_3\otimes\varphi(b_1,b_2)b_3-2\varphi(a_1,a_2)a_3\otimes[[f_1,f_2],f_3]\otimes\varphi(b_1,b_2)b_3\big)=0$$

□

Now, using again Lemma 1, it follows the simplicity of the model and then:

Corolary. *In case $\varepsilon_i=-2$ for $i=1,2,3,4,7$, $\varepsilon_i=1$ for $i=5,6$, $\gamma_i=-2$ for $i=1,3$ and $\gamma_2=2$, the above algebra $\mathcal{L}$ is simple of dimension 52, hence F_4.*

4.4 A $\mathbb{Z}_2^3$-grading of F_4

Our last subalgebra, of type $4A_1$, provides a $\mathbb{Z}_2^3$-grading of F_4, so it divides F_4 in more pieces than before, but with the advantage that all the pieces are easily handled with because of its small size, since there are only trivial and natural modules for each of the A_1's.

A precedent is the model for D_4 based in $4A_1$, which appears in [LMa2], where there is an interesting version of triality for the split algebra D_4 over $\mathbb{R}$. Take four two-dimensional vector spaces V_i, $i=1,2,3,4$, and $b_i\colon V_i\times V_i\to F$ nondegenerate forms on each of them. Then $\mathcal{L}=\sum\operatorname{sl}(V_i)\oplus(V_1\otimes V_2\otimes V_3\otimes V_4)$ is a Lie algebra of type D_4, with $\sum\operatorname{sl}(V_i)$ a Lie subalgebra acting on $\otimes V_i$ in the obvious way, and being the bracket $[\otimes v_i,\otimes w_i]=\sum_{i=1}^4\Pi_{j\neq i}b_j(v_j,w_j)v_iw_i$, by means of the natural isomorphisms $S^2V_i\to\operatorname{sl}(V_i)$. Then, the three eight-dimensional representations are $(V_1\otimes V_2)\oplus(V_3\otimes V_4)$, $(V_1\otimes V_3)\oplus(V_2\otimes V_4)$ and $(V_1\otimes V_4)\oplus(V_2\otimes V_3)$, inequivalent, where the action of $\operatorname{sl}(V_i)$ is the obvious one and the action of $\otimes V_i$ is given by $[v_1\otimes v_2\otimes v_3\otimes v_4,u_i\otimes u_j]=b_i(v_i,u_i)b_j(v_j,u_j)v_k\otimes v_l$ (i,j,k,l different indices).

Now, only one step is necessary to give a model for F_4 based on $4A_1$, taking into account the isomorphism between F_4 and D_4 plus its three three-dimensional representations: the determination of the suitable scalars.

Theorem 4. *Let V_i be a two-dimensional vector space, and $b_i\approx b\colon V_i\times V_i\to F$ a skew-symmetric bilinear form, for any $i=1,2,3,4$. We define a $\mathbb{Z}_2^3$-graded anticommutative algebra*

$$\mathcal{L}=\operatorname{sp}(V_1)\oplus\operatorname{sp}(V_2)\oplus\operatorname{sp}(V_3)\oplus\operatorname{sp}(V_4)\oplus V_1\otimes V_2\otimes V_3\otimes V_4\oplus\sum_{i<j}V_i\otimes V_j$$

such that $\mathcal{L}_{(0,0,0)}=\sum_{i=1}^4\operatorname{sp}(V_i)$, $\mathcal{L}_{(a_1,a_2,a_3)}=\otimes\{V_i\mid i=1,2,3,4,\quad a_i=1\}$, being $a_4=a_1+a_2+a_3\in\mathbb{Z}_2$, where $\mathcal{L}_{(0,0,0)}$ is a Lie subalgebra, the brackets $[\mathcal{L}_{(0,0,0)},\mathcal{L}_{(a_1,a_2,a_3)}]$ are given by the natural action as modules and

$$\begin{aligned}
&[v_1\otimes v_2\otimes v_3\otimes v_4,w_1\otimes w_2\otimes w_3\otimes w_4] &&=\textstyle\sum_{i=1}^4(\varphi_{v_i,w_i}\pi_{j\neq i}b(v_j,w_j))\\
&[v_1\otimes v_2\otimes v_3\otimes v_4,w_i\otimes w_j] &&=\gamma_{ij}b(v_i,w_i)b(v_j,w_j)v_k\otimes v_l\\
&[v_i\otimes v_j,w_i\otimes w_j] &&=\varepsilon_{ij}(b(v_i,w_i)\varphi_{v_j,w_j}+b(v_j,w_j)\varphi_{v_i,w_i})\\
&[v_i\otimes v_j,w_i\otimes w_k] &&=\beta_{il}b(v_i,w_i)v_j\otimes w_k\\
&[v_i\otimes v_j,v_k\otimes v_l] &&=\alpha_{ij}v_1\otimes v_2\otimes v_3\otimes v_4
\end{aligned}$$

for any $v_i, w_i \in V_i$, where the indices i,j,k,l are distinct, $\varphi_{u,v} \equiv b(v,-)w + b(w,-)v$, and the scalars are

$$\gamma_{ij} = \begin{cases} 2 & \text{if } i,j \neq 4 \\ -1 & i \text{ or } j = 4 \end{cases} \qquad \alpha_{ij} = \begin{cases} 1 & \text{if } i,j \neq 4 \\ -1 & i \text{ or } j = 4 \end{cases}$$

$$\varepsilon_{ij} = \begin{cases} 1 & \text{if } i,j \neq 4 \\ \frac{1}{2} & i \text{ or } j = 4 \end{cases} \qquad \beta_{ij} = \begin{cases} 1 & \text{if } i,j \neq 4 \\ -1 & j = 4 \\ \frac{1}{2} & i = 4 \end{cases}$$

Then $\mathcal{L}$ is a simple Lie algebra of dimension 52, of type F_4.

Remark. It is a straightforward computation that the Jacobi identity is verified if and only if $\gamma_{ij}\gamma_{kl} = -2$, $\gamma_{ij}\varepsilon_{kl} = \alpha_{ij}$, $\gamma_{ij}\beta_{kj} = -\gamma_{jk}\beta_{il}$, $\varepsilon_{ij} = \beta_{il}\beta_{jl}$, $\beta_{ij}\alpha_{kl} = \beta_{il}\alpha_{jk}$, and $\alpha_{ik}\gamma_{ij} = \beta_{ij}\beta_{jk} - \beta_{ji}\beta_{il}$, hence the above choice is a right possibility.

But a shorter way to choose suitable scalars is from the recent approach of the magic square due to Elduque [E2, §3.1] using symmetric composition algebras.

5 Final comments

5.1 Once described the models associated to the subalgebras, let us look at the close relationship between several of them. As in 3.3, if $F_4 = \oplus_{g\in G}\mathcal{L}_g$ is a G-grading with $\mathcal{L}_0 = \mathfrak{h}$ semisimple of rank 4, and $\tilde{G}$ is a subgroup of G, then $\mathfrak{h}_{\tilde{G}} := \oplus_{g\in\tilde{G}}\mathcal{L}_g$ is also semisimple and $F_4 = \oplus_{a+\tilde{G}\in G/\tilde{G}}\mathcal{L}_{a+\tilde{G}}$ is $G/\tilde{G}$-graded, with $\mathcal{L}_{a+\tilde{G}} = \sum_{g+\tilde{G}=a+\tilde{G}}\mathcal{L}_g$, thus the pieces in the first grading fit in this one. Obviously $\mathfrak{h}_{\{0\}} = \mathfrak{h}$ and $\mathfrak{h}_G = F_4$.

Let us examine, for example, the nontrivial subgroups for $G = \mathbb{Z}_2^3$. In this case $\mathfrak{h} = \sum \mathrm{sp}(V_i)$ and $F_4 = \sum \mathrm{sp}(V_i) \oplus V_1\otimes V_2\otimes V_3\otimes V_4 \oplus \sum_{i<j} V_i\otimes V_j$, and thus the following subalgebras are obtained related to the following subgroups:

• for $G_1 = \langle(1,1,0)\rangle$ we get $\mathfrak{h}_{G_1} = \mathcal{L}_{(0,0,0)} \oplus \mathcal{L}_{(1,1,0)} = (\mathrm{sp}(V_1)\oplus\mathrm{sp}(V_2)\oplus V_1\otimes V_2)\oplus\mathrm{sp}(V_3)\oplus\mathrm{sp}(V_4)$, of type $C_2\oplus A_1\oplus A_1$ (the grading group is $G/G_1 \approx \mathbb{Z}_2^2$);

• for $G_2 = \langle(1,1,1)\rangle$ we get $\mathfrak{h}_{G_2} = \mathcal{L}_{(0,0,0)} \oplus \mathcal{L}_{(1,1,1)} = \sum\mathrm{sp}(V_i)\oplus V_1\otimes V_2\otimes V_3\otimes V_4$, of type D_4 (the grading group is $G/G_2 \approx \mathbb{Z}_2^2$);

• for $G_3 = \{(a_1,a_2,a_3) \mid \sum_{i=1}^3 a_i = 0\}$ we get $\mathfrak{h}_{G_3} = \mathcal{L}_{(0,0,0)}\oplus\mathcal{L}_{(1,1,0)}\oplus\mathcal{L}_{(0,1,1)}\oplus\mathcal{L}_{(1,0,1)} = (\mathrm{sp}(V_1)\oplus\mathrm{sp}(V_2)\oplus\mathrm{sp}(V_3)\oplus V_1\otimes V_2\oplus V_2\otimes V_3\oplus V_1\otimes V_3)\oplus\mathrm{sp}(V_4)$, of type $C_3\oplus A_1$ (the grading group is $G/G_3 \approx \mathbb{Z}_2$);

• for $G_4 = \langle(1,1,0),(1,1,1)\rangle$ we get $\mathfrak{h}_{G_4} = \mathcal{L}_{(0,0,0)}\oplus\mathcal{L}_{(1,1,0)}\oplus\mathcal{L}_{(1,1,1)}\oplus\mathcal{L}_{(0,0,1)} = \sum\mathrm{sp}(V_i)\oplus V_1\otimes V_2\oplus V_1\otimes V_2\otimes V_3\otimes V_4\oplus V_3\otimes V_4$, of type B_4 (the grading group is $G/G_4\approx\mathbb{Z}_2$).

Consequently, the $4A_1$-construction is a refinement of five of the seven described models of F_4.

Something similar occurs for the group $\mathbb{Z}_4$. In this case $\mathfrak{h} = \mathrm{sl}(V)\oplus\mathrm{sl}(W)\approx A_1\oplus A_3$ and $F_4 = \mathfrak{h}\oplus V\otimes W\oplus S^2V\otimes\bigwedge^2 W\oplus V\otimes W^*$, and for its subgroup $\tilde{G} = \{0,2\}$, the corresponding subalgebra is $\mathfrak{h}_{\tilde{G}} = \mathrm{sl}(V)\oplus\mathrm{sl}(W)\oplus S^2V\otimes\bigwedge^2 W$, of type B_4, and its grading group is $G/\tilde{G} = \mathbb{Z}_4/\{(0,2)\}\approx\mathbb{Z}_2$, as we knew.

In particular, the only maximal semisimple subalgebras of F_4 of rank 4 are $2A_2$, $C_3\oplus A_1$ and B_4, because $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are the only simple grading groups, as we knew from 2.2 and 3.4. In Dynkin's work [Dy, table 12], the subalgebra of type $A_1\oplus A_3$ is considered maximal because of a mistake.

5.2 All the models and gradings of the exceptional Lie algebras G_2 and E_6 relative to semisimple subalgebras of maximal rank have already been described in this work.

The semisimple subalgebras of rank 2 of G_2 are only $2A_1$ and A_2, which produce a $\mathbb{Z}_2$- and a $\mathbb{Z}_3$-grading of G_2 respectively, and the semisimple subalgebras of rank 6 of E_6 are $A_1 \oplus A_5$ and $3A_2$, which again produce a $\mathbb{Z}_2$- and a $\mathbb{Z}_3$-grading of E_6 respectively. The models are described in the introduction (1) and 4.1, and they had appeared in [A, Ch 13], [BeDrE1], [E2].

As regards the remaining exceptional Lie algebras, the situation is richer. Let us put our attention in E_8. There are 14 semisimple subalgebras of rank 8, namely:

D_8, $E_7 \oplus A_1$	$\to$	$\mathbb{Z}_2$	A_8, $E_6 \oplus A_2$	$\to$	$\mathbb{Z}_3$
$D_6 \oplus 2A_1$, $2D_4$	$\to$	$\mathbb{Z}_2^2$	$4A_2$	$\to$	$\mathbb{Z}_3^2$
$2A_3 \oplus 2A_1$	$\to$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$A_5 \oplus A_2 \oplus A_1$	$\to$	$\mathbb{Z}_6$
$D_4 \oplus 4A_1$	$\to$	$\mathbb{Z}_2^3$	$2A_4$	$\to$	$\mathbb{Z}_5$
$8A_1$	$\to$	$\mathbb{Z}_2^4$	$A_7 \oplus A_1$, $D_5 \oplus A_3$	$\to$	$\mathbb{Z}_4$

where we have written too the associated grading groups according to Theorem 1. From them, the most known models of E_8 are the ones based on A_8 [FuH, §22.4], D_8 [A, Ch 7], and $2D_4$ (magic squares, for instance [E]), joined with $E_6 \oplus A_1$ and $E_7 \oplus A_1$ (1). Hence it is possible to develop several unknown models of E_8 based on subalgebras of rank 8. For instance, if V and W are 5-dimensional vector subspaces, we obtain the $\mathbb{Z}_5$-grading

$$E_8 = \mathrm{sl}(V) \oplus \mathrm{sl}(W) \oplus V \otimes \bigwedge^2 W \oplus \bigwedge^2 V \otimes W^* \oplus \bigwedge^2 V^* \otimes W \oplus V^* \otimes \bigwedge^2 W^*$$

and the products between the components are very easy to describe. Other examples using trace-zero endomorphisms and exterior powers of a vector space are the $\mathbb{Z}_6$-grading

$$\begin{aligned} E_8 = {} & \mathrm{sl}(U) \oplus \mathrm{sl}(V) \oplus \mathrm{sl}(W) \\ & \oplus U \otimes V \otimes W \oplus F \otimes V^* \otimes \textstyle\bigwedge^2 W \oplus U \otimes F \otimes \bigwedge^3 W \oplus F \otimes V \otimes \bigwedge^4 W \oplus U \otimes V^* \otimes \bigwedge^5 W \end{aligned}$$

for U, V and W vector spaces of dimensions 2, 3 and 6 respectively, and the more complicated $\mathbb{Z}_2 \times \mathbb{Z}_4$-grading

$$\begin{aligned} E_8 = {} & \mathrm{sl}(W)^2 \oplus \mathrm{sl}(V)^2 \oplus W \otimes W \otimes V \otimes F \oplus \textstyle\bigwedge^2 W \otimes \bigwedge^2 W \otimes F \otimes F \\ & \oplus W^* \otimes W^* \otimes V^* \otimes F \oplus F \otimes \textstyle\bigwedge^2 W \otimes V \otimes V \oplus W \otimes W^* \otimes F \otimes V \\ & \oplus \textstyle\bigwedge^2 W \otimes F \otimes V \otimes V \oplus W^* \otimes W \otimes F \otimes V \end{aligned}$$

for W and V spaces of dimension 4 and 2 respectively. Though the decompositions don't look very nice, the products are quite simple, in terms of tensors. It is slightly difficult if $\mathfrak{h}$ contains an ideal of type D_n, since the spin modules appear in the decomposition. The less known gradings are the $\mathbb{Z}_4$-grading

$$E_8 = D_5 \oplus \mathrm{sl}(V) \oplus V(\lambda_4) \otimes V \oplus V(\lambda_1) \otimes \bigwedge^2 V \oplus V(\lambda_5) \otimes V^*$$

for V a 4-dimensional vector space, and the $\mathbb{Z}_2^2$-grading

$$E_8 = \mathrm{sl}(U) \oplus D_6 \oplus \mathrm{sl}(V) \oplus U \otimes V(\lambda_5) \otimes F \oplus U \otimes V(\lambda_1) \otimes V \oplus F \otimes V(\lambda_6) \otimes V$$

for U and V two-dimensional vector spaces. A complete description of the products with suitable scalars has recently be exposed in [DrMar], part of a work in preparation.

5.3 There are some series of decompositions for the exceptional Lie algebras based on semisimple subalgebras of maximal rank, with unified descriptions, which allow to give models for the five exceptional Lie algebras with the same grading group.

For example, the authors in [LMa1] consider, over the complex numbers,

$$\begin{aligned} g(A) &= \mathrm{so}_8 \oplus t(A) \oplus \mathbb{O}_1 \otimes A_1 \oplus \mathbb{O}_2 \otimes A_2 \oplus \mathbb{O}_3 \otimes A_3 \\ g(A) &= \mathrm{sl}_3 \oplus \mathrm{sl}_3(A) \oplus \mathbb{C}^3 \otimes H(M_3(A)) \oplus \mathbb{C}^{3*} \otimes H(M_3(A))^* \\ g(A) &= \mathrm{sl}_2 \oplus \mathrm{sp}_6(A) \oplus \mathbb{C}^2 \otimes \bigwedge^{<3>} A^6 \end{aligned}$$

The first one provides a $\mathbb{Z}_2^2$-grading of F_4, E_6, E_7 and E_8 based on D_4, $D_4 \oplus 2F$, $D_4 \oplus 3A_1$ and $2D_4$ respectively (this case is worked out in a version of the Freudenthal magic square, so the related models are known); the second case provides a $\mathbb{Z}_3$-grading of F_4, E_6, E_7 and E_8 based on $2A_2$, $3A_2$, $A_2 \oplus A_5$ and $A_2 \oplus E_6$ respectively; and the third one a $\mathbb{Z}_2$-grading of F_4, E_6, E_7 and E_8 based on $A_1 \oplus C_3$, $A_1 \oplus A_5$, $A_1 \oplus D_6$ and $A_1 \oplus E_7$ respectively (corresponding to the construction (1)).

5.4 All our models are compatible with the root system in the following sense: if $\mathcal{L} = \oplus_{g\in G}\mathcal{L}_g$ is one of our gradings, then there is a Cartan subalgebra H with associated root decomposition $\mathcal{L} = H \oplus (\oplus_{\alpha\in\Phi} L_\alpha)$ such that, for any $g \in G$, $\mathcal{L}_g = (H \cap \mathcal{L}_g) \oplus (\oplus_{\alpha\in\Phi} L_\alpha \cap \mathcal{L}_g)$. Conversely, if the simple Lie algebra $\mathcal{L}$ is graded by a group G in a way compatible with the root system, then $\mathcal{L}_0$ is a maximal rank reductive subalgebra (it is regular and it contains H). There are some researchers interested precisely in gradings *not* compatible with the root decomposition, for example in fine gradings [P], related to contractions and physics problems [PPoW].

5.5 The decomposition of F_4 as a sum of modules for its subalgebra of type C_3 is $F_4 \approx V(2\lambda_1) \oplus 2V(\lambda_2) \oplus 3V(0)$, a grading by the root system C_3, which is usually called a Δ-grading, since $2\lambda_1$ and λ_2 are the maximal long and short roots respectively.

If $\mathfrak{h}$ is another simple regular subalgebra of F_4 and Δ is the root system of $\mathfrak{h}$, F_4 is not Δ-graded, as it is easily checked from the previous decompositions of F_4 as a sum of modules.

In fact, F_4 decomposes as a sum of $\mathfrak{h}$-submodules of types adjoint, trivial, $V(\lambda_i)$ and $V(2\lambda_i)$, for λ_i a fundamental weight. Notice that the same happens in E_8, without modules of type $V(2\lambda_i)$, so $\mathfrak{m}$ as $\mathfrak{h}$-module is sum of only basic and trivial modules.

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