LIE SUPERALGEBRAS GRADED BY THE ROOT SYSTEMS C(n), D(m,n), D(2,1;α), F(4), AND G(3)

Georgia Benkart[1]
Alberto Elduque[2]

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To Professor Robert Moody with our best wishes on his sixtieth birthday

Abstract. We determine the Lie superalgebras that are graded by the root systems of the basic classical simple Lie superalgebras of type $\mathrm{C}(n)$, $\mathrm{D}(m,n)$, $\mathrm{D}(2,1;\alpha)$ ($\alpha \in \mathbb{F}\setminus\{0,-1\}$), $\mathrm{F}(4)$, and $\mathrm{G}(3)$.

§1. Introduction

The concept of a Lie algebra graded by a finite root system was defined and investigated by Berman and Moody [BM] as an approach for studying various important classes of Lie algebras such as the intersection matrix Lie algebras of Slodowy [S], which arise in the study of singularities, or the extended affine Lie algebras of [AABGP]. The unifying theme is that these Lie algebras exhibit a grading by a finite (possibly nonreduced) root system Δ. The formal definition depends on a finite-dimensional split simple Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$ of characteristic zero having a root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\mu\in\Delta} \mathfrak{g}_\mu$ relative to a split Cartan subalgebra $\mathfrak{h}$. Such a Lie algebra $\mathfrak{g}$ is an analogue over $\mathbb{F}$ of a finite-dimensional complex simple Lie algebra.

Definition 1.1. *A Lie algebra L over $\mathbb{F}$ is* **graded by the (reduced) root system Δ** *or is* **Δ-graded** *if*

(ΔG1) *L contains as a subalgebra a finite-dimensional split simple Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\mu\in\Delta} \mathfrak{g}_\mu$ whose root system is Δ relative to a split Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_0$;*

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(ΔG2) $L = \bigoplus_{\mu\in\Delta\cup\{0\}} L_\mu$, *where* $L_\mu = \{x \in L \mid [h,x] = \mu(h)x$ *for all* $h \in \mathfrak{h}\}$ *for* $\mu \in \Delta \cup \{0\}$; *and*

(ΔG3) $L_0 = \sum_{\mu\in\Delta}[L_\mu, L_{-\mu}]$.

There is also a notion of a Lie algebra graded by the nonreduced root system BC_r introduced and studied in [ABG2] (see also [BS] for the BC_1-case). The Lie algebras graded by finite root systems (both reduced and nonreduced) decompose relative to the adjoint action of $\mathfrak{g}$ into a direct sum of finite-dimensional irreducible $\mathfrak{g}$-modules. There is one possible isotypic component corresponding to each root length and one corresponding to 0 (the sum of the trivial $\mathfrak{g}$-modules). Thus, for the simply-laced root systems only adjoint modules and trivial modules occur. For the doubly-laced root systems, copies of the module having the highest short root as its highest weight also can occur. For type BC_r, there are up to four isotypic components, except when the grading subalgebra $\mathfrak{g}$ has type $\mathrm{D}_2 \cong \mathrm{A}_1\times\ \mathrm{A}_1$, where there are five possible isotypic components. The complexity increases with the number of isotypic components. These $\mathfrak{g}$-module decompositions and the representation theory of $\mathfrak{g}$ play an essential role in the classification of the Lie algebras graded by finite root systems, which has been accomplished in the papers [BM], [BZ], [N], [ABG1], [ABG2], [BS].

Our focus here and in [BE1], [BE2] is on Lie superalgebras graded by the root systems of the finite-dimensional basic classical simple Lie superalgebras $\mathrm{A}(m,n)$, $\mathrm{B}(m,n)$, $\mathrm{C}(n)$, $\mathrm{D}(m,n)$, $\mathrm{D}(2,1;\alpha)$ ($\alpha \in \mathbb{F}\setminus\{0,-1\}$), $\mathrm{F}(4)$, and $\mathrm{G}(3)$. (A standard reference for results on simple Lie superalgebras is Kac's ground-breaking paper [K1].)

Let $\mathfrak{g}$ be a finite-dimensional split simple basic classical Lie superalgebra over a field $\mathbb{F}$ of characteristic zero with root space decomposition $\mathfrak{g} = \mathfrak{h}\oplus\bigoplus_{\mu\in\Delta}\mathfrak{g}_\mu$ relative to a split Cartan subalgebra $\mathfrak{h}$. Thus, $\mathfrak{g}$ is an analogue over $\mathbb{F}$ of a complex simple Lie superalgebra whose root system Δ is of type $\mathrm{A}(m,n)$ ($m \geq n \geq 0$, $m+n \geq 1$), $\mathrm{B}(m,n)$ ($m \geq 0$, $n \geq 1$), $\mathrm{C}(n)$ ($n \geq 3$), $\mathrm{D}(m,n)$ ($m \geq 2$, $n \geq 1$), $\mathrm{D}(2,1;\alpha)$ ($\alpha \in \mathbb{F}\setminus\{0,-1\}$), $\mathrm{F}(4)$, and $\mathrm{G}(3)$. These Lie superalgebras can be characterized by the properties of being simple, having reductive even part, and having a nondegenerate even supersymmetric bilinear form. Mimicking Definition 1.1, we say

Definition 1.2. *(Compare [BE1, Defn. 1.4] and [GN, Sec. 4.7].) A Lie superalgebra L over $\mathbb{F}$ is* **graded by the root system Δ** *or is* **Δ-graded** *if*

(i) *L contains as a subsuperalgebra a finite-dimensional split simple basic classical Lie superalgebra $\mathfrak{g} = \mathfrak{h}\oplus\bigoplus_{\mu\in\Delta}\mathfrak{g}_\mu$ whose root system is Δ relative to a split Cartan subalgebra $\mathfrak{h} = \mathfrak{g}_0$;*

(ii) *(ΔG2) and (ΔG3) of Definition 1.1 hold for L relative to the root system Δ.*

The $\mathrm{B}(m,n)$-graded Lie superalgebras were determined in [BE1]. These Lie superalgebras differ from rest because of their complicated structure and most closely resemble the Lie algebras graded by the nonreduced root systems BC_r. In this work we tackle Δ-graded Lie superalgebras for $\Delta = \mathrm{C}(n)$, $\mathrm{D}(m,n)$, $\mathrm{D}(2,1;\alpha)$ ($\alpha \in$

$\mathbb{F} \setminus \{0,-1\})$, F(4), and G(3). Our main theorem (Theorem 5.2) completely describes the structure of the Lie superalgebras graded by these root systems. The A(n,n)-graded Lie superalgebras are truly exceptional for several reasons, and their study (along with A(m,n)-graded Lie superalgebras for $m \neq n$) forms the subject of [BE2].

We would like to view a Δ-graded Lie superalgebra L as a $\mathfrak{g}$-module in order to determine its structure. However, a major obstacle encountered in the superalgebra case is that $\mathfrak{g}$-modules need not be completely reducible. We circumvent this roadblock below (and previously in [BE1]) by showing that a Δ-graded Lie superalgebra L must be completely reducible as a module for its grading subsuperalgebra $\mathfrak{g}$ in all cases except when Δ is of type A(n,n).

§2. The $\mathfrak{g}$-module structure of Δ-graded Lie superalgebras for $\Delta = \mathrm{C}(n),\ \mathrm{D}(m,n),\ \mathrm{D}(2,1;\alpha)\ (\alpha \in \mathbb{F} \setminus \{0,-1\}),\ \mathrm{F}(4)$, and $\mathrm{G}(3)$

The following result is instrumental in examining Δ-graded Lie superalgebras.

Lemma 2.1. *([BE1, Lemma 2.2]) Let L be a Δ-graded Lie superalgebra, and let $\mathfrak{g}$ be its grading subsuperalgebra. Then L is locally finite as a module for $\mathfrak{g}$.*

This result says that each element of a Δ-graded Lie superalgebra L, in particular each weight vector of L relative to the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, generates a finite-dimensional $\mathfrak{g}$-module. Such a finite-dimensional module has a $\mathfrak{g}$-composition series whose irreducible factors have weights which are roots of $\mathfrak{g}$ or 0. Next we determine which finite-dimensional irreducible $\mathfrak{g}$-modules have weights which are roots of $\mathfrak{g}$ or are 0. For this purpose, we will need to do a case-by-case analysis.

G(3) case.

When $\mathfrak{g}$ is of type G(3), its even part $\mathfrak{g}_{\bar{0}}$ is a sum of two ideals, $\mathfrak{g}_{\bar{0}} = \mathfrak{s}_1 \oplus \mathfrak{s}_2$, where $\mathfrak{s}_1$ is a simple Lie algebra type G_2 and $\mathfrak{s}_2$ is $\mathfrak{sl}_2$. We assume that $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$, where $\mathfrak{h}_2 = \mathbb{F}h$, a Cartan subalgebra of $\mathfrak{sl}_2$, and $\mathfrak{h}_1$ is a Cartan subalgebra of an $\mathfrak{sl}_3$ subalgebra of $\mathfrak{s}_1$.

As in [K1, §2.5.4], $\Delta = \Delta_{\bar{0}} \cup \Delta_{\bar{1}}$ (even and odd roots relative $\mathfrak{h}$), where

$$\begin{aligned}
&\Delta_{\bar{0}} = \{\varepsilon_i - \varepsilon_j,\ \pm\varepsilon_i \mid i \neq j,\ i,j = 1,2,3,\ \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0\} \cup \{\pm 2\delta\},\\
&\Delta_{\bar{1}} = \{\pm\varepsilon_i \pm \delta,\ \pm\delta\}, \qquad \text{and}\\
&\ \Pi = \{\alpha_1 = \delta + \varepsilon_1,\ \alpha_2 = \varepsilon_2,\ \alpha_3 = \varepsilon_3 - \varepsilon_2\}
\end{aligned} \tag{2.2}$$

is a system of simple roots. Here we suppose that $\delta(h) = 1$, and that $\mathfrak{h}_1 \subset \mathfrak{sl}_3 \subset \mathfrak{s}_1$ consists of diagonal matrices $d = \mathrm{diag}\{d_1, d_2, d_3\}$ with trace $d_1 + d_2 + d_3 = 0$, and $\varepsilon_i(d) = d_i$. We also assume that $\delta(\mathfrak{h}_1) = 0 = \varepsilon_i(\mathfrak{h}_2)$ for all i. Solving the system $\alpha_j(h_i) = a_{i,j}$, where $a_{i,j}$ is the (i,j) entry of the Cartan matrix (see p. 49 of [K1])

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & -3 \\ 0 & -1 & 2 \end{pmatrix}, \tag{2.3}$$

we obtain the coroots

$$
\begin{aligned}
h_1 &= 2h + \mathrm{diag}(-2,1,1)\\
h_2 &= \mathrm{diag}(-1,2,-1)\\
h_3 &= \mathrm{diag}(0,-1,1).
\end{aligned}
\tag{2.4}
$$

Now the conditions for $\Lambda \in \mathfrak{h}^*$ to be the highest weight of a finite-dimensional irreducible $\mathfrak{g}$-module $V(\Lambda)$ are given in [K1, Thm. 8] or [K2, Prop. 2.3] in terms of the values $\Lambda(h_i) = a_i$. For G(3) they are

(2.5)

(i) a_2 and $a_3 \in \mathbb{Z}_{\geq 0}$;
(ii) $k = \frac{1}{2}\big(a_1 - 2a_2 - 3a_3\big) \in \mathbb{Z}_{\geq 0}$ and $k \neq 1$;
(iii) If $k = 0$, then all $a_i = 0$, (i.e. $\Lambda = 0$); and if $k = 2$, then $a_2 = 0$.

The roots that satisfy constraints (i) and (ii) are $\varepsilon_3 - \varepsilon_1$ (the highest long root of G_2), $-\varepsilon_1$ (the highest short root of G_2), and 2δ (the positive root of $\mathfrak{sl}_2$ and the highest root of G(3)). (Note that δ satisfies (i) but has $k = 1$.) Both $\Lambda = \varepsilon_3 - \varepsilon_1$ and $\Lambda = -\varepsilon_1$ have $k = 0$ so they can be ruled out. Thus, the only finite-dimensional irreducible modules having weights that are roots or 0 are the adjoint module (with highest weight 2δ) or the trivial module. We allow the possibility that the highest weight vector in these modules has its parity changed from even to odd.

F(4) case.

When $\mathfrak{g}$ is of type F(4), its even part is a sum of two ideals, $\mathfrak{g}_{\bar 0} = \mathfrak{s}_1 \oplus \mathfrak{s}_2$, where $\mathfrak{s}_1$ is a simple Lie algebra type B_3 and $\mathfrak{s}_2$ is $\mathfrak{sl}_2$. We assume that $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$, where $\mathfrak{h}_2 = \mathbb{F}h$, a Cartan subalgebra of $\mathfrak{sl}_2$, and $\mathfrak{h}_1$ is a Cartan subalgebra of $\mathfrak{s}_1$ (which we identify with the orthogonal Lie algebra $\mathfrak{o}_7$).

As in [K1, §2.5.4],

$$
\begin{aligned}
\Delta_{\bar 0} &= \{\pm\varepsilon_i \pm \varepsilon_j,\ \pm\varepsilon_i \mid i \neq j,\ i,j = 1,2,3\} \cup \{\pm\delta\},\\
\Delta_{\bar 1} &= \left\{\frac{1}{2}\big(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \delta\big)\right\}, \qquad \text{and}\\
\Pi &= \left\{\alpha_1 = \frac{1}{2}\big(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \delta\big),\ \alpha_2 = -\varepsilon_1,\ \alpha_3 = \varepsilon_1 - \varepsilon_2,\ \alpha_4 = \varepsilon_2 - \varepsilon_3\right\}
\end{aligned}
\tag{2.6}
$$

is a system of simple roots. Here we suppose that $\delta(h) = 2$, and that $\mathfrak{h}_1 \subset \mathfrak{s}_1$ consists of diagonal matrices $d = \mathrm{diag}\{0, d_1, d_2, d_3, -d_1, -d_2, -d_3\}$ with $\varepsilon_i(d) = d_i$. We also assume that $\delta(\mathfrak{h}_1) = 0 = \varepsilon_i(\mathfrak{h}_2)$ for all i. Let $t_1 = \mathrm{diag}\{0,1,0,0,-1,0,0\}$, $t_2 = \mathrm{diag}\{0,0,1,0,0,-1,0\}$, and $t_3 = \mathrm{diag}\{0,0,0,1,0,0,-1\}$. Then solving the system $\alpha_j(h_i) = a_{i,j}$ coming from the Cartan matrix

$$
\begin{pmatrix}
0 & 1 & 0 & 0\\
-1 & 2 & -2 & 0\\
0 & -1 & 2 & -1\\
0 & 0 & -1 & 2
\end{pmatrix},
\tag{2.7}
$$

we obtain the coroots

$$
\begin{aligned}
h_1 &= -t_1 - t_2 - t_3 + \frac{3}{2}h \\
h_2 &= -2t_1 \\
h_3 &= t_1 - t_2 \\
h_4 &= t_2 - t_3.
\end{aligned} \tag{2.8}
$$

Here the conditions for $\Lambda \in \mathfrak{h}^*$ to be the highest weight of a finite-dimensional irreducible $\mathfrak{g}$-module $V(\Lambda)$ are, in terms of the values $\Lambda(h_i) = a_i$, given by

(2.9)

(i) a_2, a_3, and $a_4 \in \mathbb{Z}_{\geq 0}$;
(ii) $k = \frac{1}{3}\big(2a_1 - 3a_2 - 4a_3 - 2a_4\big) \in \mathbb{Z}_{\geq 0}$ and $k \neq 1$;
(iii) If $k = 0$, then all $a_i = 0$; if $k = 2$, then $a_2 = 0 = a_4$; if $k = 3$, then $a_2 = a_4 + 1$.

Only the roots $-\varepsilon_2 - \varepsilon_3$, $-\varepsilon_3$, δ, and $-\frac{1}{2}(-\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \delta)$ satisfy (i), and for each of them except δ, the corresponding value of k is 0. For $\Lambda = \delta$ (the highest root of $\mathfrak{g}$), the value of k is 2 and $a_2 = 0 = a_4$, so that all conditions hold. Thus, again the only finite-dimensional irreducible modules having weights that are roots or 0 are the adjoint module (with highest weight δ) or the trivial module and parity changes of them.

$\mathbf{D}(2,1;\alpha)$ case.

For a simple Lie superalgebra $\mathfrak{g}$ of type $\mathrm{D}(2,1;\alpha)$ ($\alpha \in \mathbb{F} \setminus \{0,-1\}$), the even part $\mathfrak{g}_{\bar 0} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 = \mathfrak{sl}_2 \otimes_{\mathbb{F}} \mathbb{F}^3$. We identify $\mathbb{F}^3$ with triples $\xi = (\xi_1, \xi_2, \xi_3)$, and the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ with $h \otimes \mathbb{F}^3$, where $\mathbb{F}h$ is the Cartan subalgebra of $\mathfrak{sl}_2$. Let $\varepsilon_i(h \otimes \xi) = \xi_i$ for $i = 1,2,3$. Then the even and odd roots and simple roots are

$$
\begin{aligned}
\Delta_{\bar 0} &= \{\pm 2\varepsilon_i, \mid i = 1,2,3\}, \\
\Delta_{\bar 1} &= \{\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3\}, \\
\Pi &= \{\alpha_1 = -(\varepsilon_1 + \varepsilon_2 + \varepsilon_3),\ \alpha_2 = 2\varepsilon_2,\ \alpha_3 = 2\varepsilon_3\}.
\end{aligned} \tag{2.10}
$$

Using the Cartan matrix

$$
\begin{pmatrix} 0 & 1 & \alpha \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}, \tag{2.11}
$$

we determine that the coroots are

$$
\begin{aligned}
h_1 &= h \otimes \frac{1}{2}\big(-(1+\alpha), 1, \alpha\big) \\
h_2 &= h \otimes (0,1,0) \\
h_3 &= h \otimes (0,0,1).
\end{aligned} \tag{2.12}
$$

By [K2, Prop. 2.3], a root Λ gives a finite-dimensional $\mathfrak{g}$-module when the values $\Lambda(h_i) = a_i$ satisfy the conditions,

(2.13)

(i) a_2 and $a_3 \in \mathbb{Z}_{\geq 0}$;

(ii) $k = \dfrac{1}{1+\alpha}\big(2a_1 - a_2 - \alpha a_3\big) \in \mathbb{Z}_{\geq 0}$;

(iii) If $k = 0$, then all $a_i = 0$; and if $k = 1$, then $(a_3 + 1)\alpha = \pm(a_2 + 1)$.

The only roots for which (i) and (ii) hold are $2\varepsilon_2$, $2\varepsilon_3$, $-\varepsilon_1+\varepsilon_2+\varepsilon_3$ and $-2\varepsilon_1$ (which is the highest root of $\mathfrak{g}$). But for the first two, $k = 0$. Now when $\Lambda = -\varepsilon_1+\varepsilon_2+\varepsilon_3$, $k = 1$, and (iii) says that $2\alpha = \pm 2$ must be true. But α is assumed to be different from 0 and -1. When $\alpha = 1$, the Lie superalgebra D(2,1;α) is isomorphic to D(2, 1). (We consider this next as part of the general D(m, n) case.) Hence for D(2, 1; α) with $\alpha \neq 0, \pm 1$, the only finite-dimensional irreducible modules with weights that are roots are the adjoint and trivial modules (and parity changes of them).

D(m, n) ($m \geq 2$, $n \geq 1$) case.

Let $V = V_{\bar 0} \oplus V_{\bar 1}$ be a $\mathbb{Z}_2$-graded vector space over a field $\mathbb{F}$ of characteristic zero, with $\dim V_{\bar 0} = 2m$ and $\dim V_{\bar 1} = 2n$, where $m \geq 2$ and $n \geq 1$. We assume $(\,|\,)$ is a nondegenerate supersymmetric bilinear form of maximal Witt index on V. Thus, we may suppose there is a basis $\{u_1, \dots, u_{2m}\}$ of $V_{\bar 0}$ and a basis $\{v_1, \dots, v_{2n}\}$ of $V_{\bar 1}$ such that

$$
\begin{aligned}
(u_i \mid u_{i+m}) &= 1 = (u_{i+m} \mid u_i) \qquad (i = 1, \dots, m)\\
(v_j \mid v_{j+n}) &= 1 = -(v_{j+n} \mid v_j) \qquad (j = 1, \dots, n),
\end{aligned}
\tag{2.14}
$$

and all other products are 0.

The space $\mathrm{End}_{\mathbb{F}}(V)$ of transformations on V inherits a $\mathbb{Z}_2$-grading: $\mathrm{End}_{\mathbb{F}}(V) = \big(\mathrm{End}_{\mathbb{F}}(V)\big)_{\bar 0} \oplus \big(\mathrm{End}_{\mathbb{F}}(V)\big)_{\bar 1}$ where $x.u \in V_{a+b}$ (subscripts read mod 2) whenever $x \in \big(\mathrm{End}_{\mathbb{F}}(V)\big)_a$ and $u \in V_b$. Setting

(2.15)
$$
\begin{aligned}
&\mathfrak{g} = \{x \in \mathrm{End}_{\mathbb{F}}(V) \mid (x.u \mid v) = -(-1)^{\bar x \bar u}(u \mid x.v) \text{ for all } u, v \in V\},\\
&\mathfrak{s} = \{s \in \mathrm{End}_{\mathbb{F}}(V) \mid (s.u \mid v) = (-1)^{\bar s \bar u}(u \mid s.v) \text{ for all } u, v \in V \text{ and } \mathfrak{str}(s) = 0\},
\end{aligned}
$$

we have that $\mathfrak{g}$ is the orthosymplectic split simple Lie superalgebra $\mathfrak{osp}_{2m,2n}$ of type D(m, n). (In displays such as (2.15), we assume all elements shown are homogeneous, and our convention is that $\bar u = b$ (viewed as an element of $\mathbb{Z}_2$) whenever $u \in V_b$.) The transformations $s \in \mathfrak{s}$ are supersymmetric relative to the form on V and have supertrace 0. Thus, $\mathfrak{str}(s) = \mathfrak{tr}_{V_{\bar 0}}(s) - \mathfrak{tr}_{V_{\bar 1}}(s) = 0$ whenever $s \in \big(\mathrm{End}_{\mathbb{F}}(V)\big)_{\bar 0}$, and the supertrace is automatically 0 for all transformations in $\big(\mathrm{End}_{\mathbb{F}}(V)\big)_{\bar 1}$. The space $\mathfrak{s}$ is an irreducible $\mathfrak{g}$-module under the natural action.

Using the basis in (2.14), we may identify linear transformations with their matrices. The diagonal matrices in $\mathfrak{g}$ form a Cartan subalgebra $\mathfrak{h}$. The corresponding even and odd roots and a system of simple roots of $\mathfrak{g}$ are given by [K1, §2.5]:

(2.16)
$$
\begin{aligned}
\Delta_{\bar 0} &= \{\pm\varepsilon_i \pm \varepsilon_j, \pm\delta_r \pm \delta_s, \pm 2\delta_r \mid 1 \leq i < j \leq m,\ 1 \leq r < s \leq n\},\\
\Delta_{\bar 1} &= \{\pm\varepsilon_i \pm \delta_r \mid 1 \leq i \leq m,\ 1 \leq r \leq n\},\\
\Pi &= \{\delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n, \delta_n - \varepsilon_1, \varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_{m-1} + \varepsilon_m\},
\end{aligned}
$$

where for any $h = \operatorname{diag}(b_1, \dots, b_m, -b_1, \dots, -b_m, c_1, \dots, c_n, -c_1, \dots, -c_n) \in \mathfrak{h}$, $\varepsilon_i(h) = b_i$ and $\delta_r(h) = c_r$ for any i, r. The corresponding Cartan matrix is

$$\left(\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & A_{n-1} & & & \vdots & & 0 & & \\
 & & & & 0 & & & & \\
 & & & & -1 & & & & \\
0 & \dots & 0 & -1 & 0 & 1 & 0 & \dots & 0\\
 & & & & -1 & & & & \\
 & & & & 0 & & & & \\
 & 0 & & & \vdots & & D_m & & \\
 & & & & 0 & & & &
\end{array}\right) \tag{2.17}$$

for $m \geq 3$ (if $n = 1$, it is just the $(m+1) \times (m+1)$ bottom right corner above), where

$$A_{n-1} = \begin{pmatrix}
2 & -1 & & & \\
-1 & & & & \\
 & & \ddots & & \\
 & & & -1 & \\
 & & -1 & 2 & -1\\
 & & & -1 & 2
\end{pmatrix} \qquad \text{and}$$

$$D_m = \begin{pmatrix}
2 & -1 & & & & \\
-1 & & & & & \\
 & & \ddots & & & \\
 & & & -1 & & \\
 & & -1 & 2 & -1 & -1\\
 & & & -1 & 2 & 0\\
 & & & -1 & 0 & 2
\end{pmatrix};$$

while for $m = 2$, the Cartan matrix is

$$\left(\begin{array}{ccccccc}
 & & & & 0 & & \\
 & A_{n-1} & & & \vdots & 0 & \\
 & & & & 0 & & \\
 & & & & -1 & & \\
0 & \dots & 0 & -1 & 0 & 1 & 1\\
 & & & & -1 & 2 & 0\\
 & 0 & & & -1 & 0 & 2
\end{array}\right). \tag{2.17'}$$

Let $t_1, \dots, t_{n+m} \in \mathfrak{h}$ be the dual basis to $\delta_1, \dots, \delta_n, \varepsilon_1, \dots, \varepsilon_m$. Then relative to this basis of $\mathfrak{h}$, the coroots $h_1, \dots, h_{n+m}$ have the following expressions:

$$\begin{aligned}
h_i &= t_i - t_{i+1} && (1 \leq i \leq n-1)\\
h_n &= t_n + t_{n+1} &&\\
h_{n+j} &= t_{n+j} - t_{n+j+1} && (1 \leq j \leq m-1)\\
h_{n+m} &= t_{n+m-1} + t_{n+m}.
\end{aligned}$$

Now, suppose

$$\Lambda = \sum_{i=1}^{n} \pi_i \delta_i + \sum_{j=1}^{m} \mu_j \varepsilon_j,$$

and $\Lambda(h_i) = a_i$ in Kac's notation. The conditions for Λ to be the highest weight of a finite-dimensional irreducible module are given in [K1, Thm. 8]:

(2.18)

(i) $a_i \in \mathbb{Z}_{\geq 0}$ for $i \neq n$;
(ii) $k = a_n - \big(a_{n+1} + \dots a_{n+m-2} + \frac{1}{2}(a_{n+m-1} + a_{n+m})\big) \in \mathbb{Z}_{\geq 0}$;
(iii) If $k \leq m-2$, then $a_{n+k+1} = \cdots = a_{n+m} = 0$; and if $k = m-1$, then $a_{n+m-1} = a_{n+m}$.

The first condition in (2.18) says

$$\begin{aligned} \pi_i - \pi_{i+1} &= a_i \in \mathbb{Z}_{\geq 0} \qquad & i = 1, \dots, n-1 \\ \mu_j - \mu_{j+1} &= a_{n+j} \in \mathbb{Z}_{\geq 0} \qquad & j = 1, \dots, m-1 \\ \mu_{m-1} + \mu_m &= a_{n+m} \in \mathbb{Z}_{\geq 0}. \end{aligned}$$

The second requirement is $\pi_n = a_n - \big(a_{n+1} + \dots a_{n+m-2} + \frac{1}{2}(a_{n+m-1} + a_{n+m})\big) = k \in \mathbb{Z}_{\geq 0}$. These two conditions imply that $\pi_1 \geq \pi_2 \geq \cdots \geq \pi_n \geq 0$ is a partition and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{m-1} \geq |\mu_m|$, with $\mu_i \in \frac{1}{2}\mathbb{Z}$ for any $i = 1, \dots, m$ (compare the results of [LS]).

The final condition is that when $k = \pi_n \leq m-2$, $\mu_{k+1} = \cdots = \mu_m = 0$; while if $k = \pi_n = m-1$, $\mu_m = 0$. Hence both cases can be combined to say that when $\pi_n \leq m-1$, then $\mu_{k+1} = \cdots = \mu_m = 0$.

If $\Lambda \in \Delta_{\bar 0} \cup \Delta_{\bar 1}$, then $\pi_n = 0$, 1 or 2, and the three conditions above imply that for $n \geq 2$, Λ is either $2\delta_1$ or $\delta_1 + \delta_2$; while for $n = 1$, Λ is either $2\delta_1$ or $\delta_1 + \varepsilon_1$. But $2\delta_1$ is the highest root, so $V(2\delta_1)$ is the adjoint module. The root $\delta_1 + \delta_2$ if $n \geq 2$ or $\delta_1 + \varepsilon_1$ if $n = 1$ is the highest weight of $\mathfrak{s}$ in (2.15). However, $2\varepsilon_1$ is a weight of $\mathfrak{s}$ which is not a root. Thus, again only the adjoint and trivial modules appear.

C(n) ($n \geq 3$) case.

The simple Lie superalgebra $\mathfrak{g}$ of type C(n) may be identified with the orthosymplectic Lie superalgebra $\mathfrak{osp}_{2,2(n-1)}$. (The restriction $n \geq 3$ comes from the isomorphism $\mathfrak{osp}_{2,2} \cong \mathfrak{sl}_{2,1}$. Thus, C(2)-graded superalgebras are regarded as A(1,0)-graded superalgebras and are described in [BE2].) For simplicity of notation, take $r = n-1$ so that $\mathfrak{g} = \mathfrak{osp}_{2,2r}$, and suppose in what follows that $r \geq 2$. We make the same identifications as for D(m,n), but here $m = 1$, and as above assume the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ consists of the diagonal matrices

$$h = \mathrm{diag}(\mu, -\mu, d, -d) \tag{2.19}$$

where $\mu \in \mathbb{F}$, and $d = \mathrm{diag}\{d_1, \dots, d_r\}$ is a diagonal matrix with entries in $\mathbb{F}$. Now for C($r+1$) = C(n):

$$\begin{aligned} \Delta_{\bar 0} &= \{\pm 2\delta_i,\ \pm\delta_i \pm \delta_j \mid 1 \leq i \neq j \leq r\} \\ \Delta_{\bar 1} &= \{\pm\varepsilon \pm \delta_i \mid 1 \leq i \leq r\}, \qquad \text{and} \\ \Pi &= \{\alpha_0 = \varepsilon + \delta_1,\ \alpha_i = \delta_i - \delta_{i+1}, (1 \leq i \leq r-1),\ \alpha_r = 2\delta_r\} \end{aligned} \tag{2.20}$$

is a system of simple roots. If h is as in (2.19), then $\varepsilon(h)=\mu$, and $\delta_i(h)=d_i$ for $i=1,\dots,n$. The corresponding Cartan matrix is

$$\begin{pmatrix} 0 & 1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & & & & \\ & & & \ddots & & & \\ & & & & -1 & 2 & -2 \\ & & & & & -1 & 2 \end{pmatrix}, \tag{2.21}$$

and the corresponding coroots ($\alpha_j(h_i)=a_{i,j}$) are given as follows (note that the row and column indices here are $-1, 0, \dots, 2r$):

$$\begin{aligned} h_0 &= (E_{-1,-1}-E_{0,0})+(E_{1,1}-E_{r+1,r+1}) \\ h_i &= (E_{i,i}-E_{r+i,r+i})-(E_{i+1,i+1}-E_{r+i+1,r+i+1}) \qquad (1\le i\le r-1) \\ h_r &= E_{r,r}-E_{2r,2r}. \end{aligned} \tag{2.22}$$

In order for $\Lambda\in\mathfrak{h}^*$ to correspond to a finite-dimensional irreducible module $V(\Lambda)$, we must have $\Lambda(h_i)\in\mathbb{Z}_{\ge 0}$ for all $i=1,\dots,r$ and $\Lambda(h_0)\in\mathbb{Z}$. Consideration of the roots in (2.20) shows that only $\Lambda=2\delta_1$, $\delta_1+\delta_2$, $-\varepsilon+\delta_1$, and $\varepsilon+\delta_1$ (the highest root of $\mathfrak{g}$) are possible solutions.

Now the Lie superalgebra $\mathfrak{g}$ has a $\mathbb{Z}$-gradation, $\mathfrak{g}=\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$ with $\mathfrak{g}_{\bar 0}=\mathfrak{g}_0$ and $\mathfrak{g}_{\bar 1}=\mathfrak{g}_{-1}\oplus\mathfrak{g}_1$. Kac [K2, Sec. 2] shows that for a finite-dimensional irreducible $\mathfrak{g}$-module $V=V(\Lambda)$, $V'=\{x\in V \mid \mathfrak{g}_1.x=0\}$ is an irreducible $\mathfrak{g}_0$-submodule of highest weight Λ, and V is a quotient of the induced module $\mathfrak{U}(\mathfrak{g})\otimes_{\mathfrak{U}(\mathfrak{g}_0\oplus\mathfrak{g}_1)}V'$, which as a vector space is isomorphic to $\mathfrak{U}(\mathfrak{g}_{-1})\otimes_{\mathbb{F}}V'$ (where $\mathfrak{U}(\ )$ denotes the universal enveloping algebra). Thus, the weights of V are of the form $\omega+\nu$, where ω is a weight of the $\mathfrak{g}_0$-module V' and ν is a weight of $\mathfrak{U}(\mathfrak{g}_{-1})$. Hence ν is either 0 or a sum of roots of the form $-\varepsilon\pm\delta_i$.

Assume that Λ is either $2\delta_1$, $\delta_1+\delta_2$, or $-\varepsilon+\delta_1$. Then with $c=E_{-1,-1}-E_{0,0}$, $(\omega+\nu)(c)\in\mathbb{Z}_{\le 0}$. But if V is a finite-dimensional module, the supertrace of the action of c is 0, so it must be $(\omega+\nu)(c)=0$ for any weight $\omega+\nu$ of V. This forces $V=V'$, so $\mathfrak{g}_1.V=0$, a contradiction since $\mathfrak{g}$ is simple, and hence V is a faithful module. Therefore, the only possibility left is $\Lambda=\varepsilon+\delta_1$, so V is the adjoint module.

§3. Complete reducibility

Proposition 3.1. *Let $\mathfrak{g}$ be one of the split simple Lie superalgebras* C(n) $(n\ge 3)$, D(m,n) $(m\ge 2,\ n\ge 1)$, D$(2,1;\alpha)$ $(\alpha\in\mathbb{F}\setminus\{0,-1\})$, F$(4)$, *or* G$(3)$ *with split Cartan subalgebra $\mathfrak{h}$. Assume V is a locally finite $\mathfrak{g}$-module satisfying*

- (i) *$\mathfrak{h}$ acts semisimply on V;*
- (ii) *any composition factor of any finite-dimensional submodule of V is isomorphic to the adjoint module $\mathfrak{g}$ or to a trivial module (possibly with the parity changed).*

Then V is a completely reducible $\mathfrak{g}$-module.

Proof. Assume X is a submodule of V, and Y is a submodule of X such that Y and X/Y are trivial or adjoint modules. By the diagonalizability of the action of $\mathfrak{h}$ on X, if X/Y and Y are isomorphic (possibly with a change in parity) with highest weight μ, then there are linearly independent weight vectors x_μ, $y_\mu \in X_\mu$ so that $X = \mathfrak{U}(\mathfrak{g})x_\mu + \mathfrak{U}(\mathfrak{g})y_\mu$. But $\mathfrak{U}(\mathfrak{g})x_\mu$ and $\mathfrak{U}(\mathfrak{g})y_\mu$ are strictly contained in X (the dimension of their highest weight spaces is 1), and both X/Y and Y are irreducible. The only possibility is that both submodules are irreducible and that $X = \mathfrak{U}(\mathfrak{g})x_\mu \oplus \mathfrak{U}(\mathfrak{g})y_\mu$, so that X is completely reducible (this is the same argument used in the proof of Theorem 3.3 of [BE1]).

As a result, it suffices to show that if Y is an adjoint module and X/Y is trivial, or if Y is trivial and X/Y is adjoint, then $X \cong Y \oplus X/Y$. When $\mathfrak{g}$ is of type $\mathrm{C}(n)$, $\mathrm{F}(4)$, or $\mathrm{G}(3)$, its Killing form is nondegenerate and $\dim \mathfrak{g}_{\bar 0} \neq \dim \mathfrak{g}_{\bar 1}$. Therefore in this case, the supertrace of the Casimir element is $\dim \mathfrak{g}_{\bar 0} - \dim \mathfrak{g}_{\bar 1} \neq 0$. Hence the Casimir element acts nontrivially on the adjoint module, and X is the direct sum of the two different eigenspaces for the Casimir element.

Now in all the remaining cases, $\mathfrak{g}_{\bar 1}$ is an irreducible module for $\mathfrak{g}_{\bar 0}$, which is a semisimple Lie algebra. In addition, $\mathrm{Hom}_{\mathfrak{g}_{\bar 0}}(\mathfrak{g}_{\bar 0} \otimes \mathfrak{g}_{\bar 1}, \mathbb{F}) = 0$, and $\mathrm{Hom}_{\mathfrak{g}_{\bar 0}}(\mathfrak{g}_{\bar 1} \otimes \mathfrak{g}_{\bar 1}, \mathbb{F})$ is spanned by a nondegenerate skew-symmetric bilinear form.

Assume initially that Y is an adjoint module. Changing the parity of X if necessary, we may assume that there is an even isomorphism of $\mathfrak{g}$-modules $\varphi : \mathfrak{g} \to Y$. By complete reducibility for $\mathfrak{g}_{\bar 0}$-modules, $X = Y \oplus \mathbb{F}v$ for some $0 \neq v \in V$ with $\mathfrak{g}_{\bar 0}.v = 0$. If $\mathfrak{g}_{\bar 1}.v \neq 0$, then by the irreducibility of $\mathfrak{g}_{\bar 1}$, we may scale v so that $x.v = \varphi(x)$ for any $x \in \mathfrak{g}_{\bar 1}$. But then for any $x, y \in \mathfrak{g}_{\bar 1}$,

$$\begin{aligned} 0 = [x,y].v &= x.(y.v) + y.(x.v) = x.\varphi(y) + y.\varphi(x) \\ &= \varphi([x,y]) + \varphi([y,x]) = 2\varphi([x,y]) \end{aligned}$$

so that $\varphi(\mathfrak{g}_{\bar 0}) = \varphi([\mathfrak{g}_{\bar 1}, \mathfrak{g}_{\bar 1}]) = 0$, a contradiction.

Finally, suppose that Y is trivial and X/Y is adjoint. As X is a completely reducible $\mathfrak{g}_{\bar 0}$-module, $X = \mathbb{F}v \oplus Z$ where $\mathfrak{g}_{\bar 0}.v = 0$ and $\mathfrak{g}_{\bar 0}.Z \neq 0$. Again we may assume that there is an even isomorphism $\psi : \mathfrak{g} \to Z$ of $\mathfrak{g}_{\bar 0}$-modules. If Z is not a $\mathfrak{g}$-submodule of X, then v is odd, and for any $x, y \in \mathfrak{g}_{\bar 1}$ and $z \in \mathfrak{g}_{\bar 0}$, $x.\psi(y) = \psi([x,y]) + (x \mid y)v$, where $(\mid)$ is a skew-symmetric form spanning $\mathrm{Hom}_{\mathfrak{g}_{\bar 0}}(\mathfrak{g}_{\bar 1} \otimes \mathfrak{g}_{\bar 1}, \mathbb{F})$, and $x.\psi(z) = \psi([x,z])$. Hence

$$\begin{aligned} \psi([[x,y],z]) &= [x,y].\psi(z) = x.(y.\psi(z)) + y.(x.\psi(z)) \\ &= x.\psi([y,z]) + y.\psi([x,z]) \\ &= \psi([x,[y,z]] + [y,[x,z]]) + \big((x \mid [y,z]) + (y \mid [x,z])\big)v \\ &= \psi([[x,y],z]) + 2(x \mid [y,z])v \end{aligned}$$

so that $(\mathfrak{g}_{\bar 1} \mid \mathfrak{g}_{\bar 1}) = (\mathfrak{g}_{\bar 1} \mid [\mathfrak{g}_{\bar 0}, \mathfrak{g}_{\bar 1}]) = 0$. We have arrived at a contradiction, so it must be that Z is a $\mathfrak{g}$-submodule of X. $\square$

§4. The structure of Lie superalgebras with certain $\mathfrak{g}$-module decompositions

From Proposition 3.1 it follows that every Lie superalgebra graded by the root system $\mathrm{C}(n)$ $(n \geq 3)$, $\mathrm{D}(m,n)$ $(m \geq 2,\, n \geq 1)$, $\mathrm{D}(2,1;\alpha)$ $(\alpha \in \mathbb{F} \setminus \{0,-1\})$, $\mathrm{F}(4)$, or $\mathrm{G}(3)$ decomposes as a $\mathfrak{g}$-module into a direct sum of adjoint modules and trivial modules. The next general result describes the structure of Lie superalgebras L having such decompositions. The restrictions imposed on L in the next lemma will hold in particular in the Δ-graded case.

Lemma 4.1. *Let L be a Lie superalgebra over $\mathbb{F}$ with a subsuperalgebra $\mathfrak{g}$, and assume that under the adjoint action of $\mathfrak{g}$, L is a direct sum of*

(1) *copies of the adjoint module $\mathfrak{g}$,*
(2) *copies of the trivial module $\mathbb{F}$.*

Assume that

(1') $\dim \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g}) = 1$ *so that* $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$ *is spanned by* $x \otimes y \mapsto [x,y]$.
(2') $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes \mathfrak{g}, \mathbb{F}) = \mathbb{F}\kappa$, *where κ is even, nondegenerate and supersymmetric,*

and the following conditions hold:

(i) *There exist $f, g \in \mathfrak{g}_{\bar 0}$ such that $[f,g] \neq 0$ and $\kappa(f,g) \neq 0$;*
(ii) *There exist $f, g, h \in \mathfrak{g}_{\bar 0}$ such that $[f,h] = [g,h] = 0$; and $\kappa(f,h) = \kappa(g,h) = 0 \neq \kappa(f,g)$,*
(iii) *There exists $f, g, h \in \mathfrak{g}_{\bar 0}$ such that $[[f,g],h] = 0 \neq [[g,h],f]$.*

Then there exist superspaces A and D such that $L \cong (\mathfrak{g} \otimes A) \oplus D$ and

(a) *A is a unital (super)commutative associative $\mathbb{F}$-superalgebra;*
(b) *D is a trivial $\mathfrak{g}$-module and is a Lie superalgebra;*
(c) *Multiplication in L is given by*

$$
\begin{aligned}
&[f \otimes a, g \otimes a'] = (-1)^{\bar a \bar g}\Big([f,g] \otimes aa' + \kappa(f,g)\langle a, a'\rangle\Big) \\
&[d, f \otimes a] = (-1)^{\bar d \bar f} f \otimes da, \\
&[d, d'] \quad \textit{(is the product in } D\textit{)}
\end{aligned}
$$

for all $f, g \in \mathfrak{g}$, $a, a' \in A$, $d, d' \in D$, where

- $\langle\, ,\, \rangle : A \times A \to D$, $(a,a') \mapsto \langle a, a'\rangle$ *is $\mathbb{F}$-bilinear, even and superskew-symmetric,*
- $[d, \langle a, a'\rangle] = \langle da, a'\rangle + (-1)^{\bar d \bar a}\langle a, da'\rangle$ *holds for $d \in D$ and $a, a' \in A$. In particular, $\langle A, A\rangle$ is an ideal of D.*
- $\Phi : D \to \mathrm{Der}_{\mathbb{F}}(A), d \mapsto \Phi(d)$ *where $\Phi(d) : a \to da$ is a representation with $\langle A, A\rangle \subseteq \ker \Phi$.*
- $0 = \sum_{\circlearrowleft} (-1)^{\bar a_1 \bar a_3}\langle a_1 a_2, a_3\rangle = 0$ *for any $a_1, a_2, a_3 \in A$.*

Conversely, the conditions above are sufficient to guarantee that a superspace $L = (\mathfrak{g} \otimes A) \oplus D$ satisfying (a)–(c) is a Lie superalgebra.

Proof. When a Lie superalgebra L is a direct sum of copies of adjoint modules and trivial modules for $\mathfrak{g}$ (allowing for changes in their parity), then after collecting isomorphic summands, we may assume there are superspaces $A = A_{\bar 0} \oplus A_{\bar 1}$ and $D = D_{\bar 0} \oplus D_{\bar 1}$ so that $L = (\mathfrak{g} \otimes A) \oplus D$. Suppose such a superalgebra L satisfies conditions (1),(2),(1)', and (2)'. Notice first that D is a subsuperalgebra of L, since it is the (super)centralizer of $\mathfrak{g}$. Fixing basis elements $\{a_i\}_{i\in I}$ of A and choosing a_i, a_j, a_k with $i, j, k \in I$, we see that the projection of the product $[\mathfrak{g} \otimes a_i, \mathfrak{g} \otimes a_j]$ onto $\mathfrak{g} \otimes a_k$ determines an element of $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$, which is spanned by the supercommutator on $\mathfrak{g}$. Thus, there exist scalars $\xi^k_{i,j}$ so that

$$[x \otimes a_i, y \otimes a_j]\Big|_{\mathfrak{g}\otimes A} = \sum_{k\in I} \xi^k_{i,j}[x, y] \otimes a_k = [x, y] \otimes \left(\sum_{k\in I} \xi^k_{i,j} a_k\right).$$

Defining $A \times A \to A$ by $a_i \times a_j \mapsto \sum_{k\in I} \xi^k_{i,j} a_k$ and extending it bilinearly, we have a product on A. Necessarily this multiplication is supercommutative because the products on $\mathfrak{g}$ and L are superanticommutative. By similar arguments (compare [BZ]), there exist bilinear pairings $A\times A \to D,\ a\times a' \mapsto \langle a, a'\rangle \in D$, and $D\times A \to A$, $d \times a \mapsto da \in A$, such that the multiplication in L is as in (c).

Now the Jacobi superidentity $\mathcal{J}(z_1, z_2, z_3) = \sum_{\circlearrowleft} (-1)^{\bar z_1 \bar z_3}[[z_1, z_2], z_3] = 0$ (cyclic permutation of the homogeneous elements z_1, z_2, z_3), when specialized with homogeneous elements $d_1,\, d_2 \in D$ and $f \otimes a \in \mathfrak{g} \otimes A$, and then with $d \in D$ and $f \otimes a,\, g \otimes a' \in \mathfrak{g} \otimes A$ will show that $\Phi(d)a = da$ is a representation of D as superderivations of A. We assume next that f, g are taken to satisfy (i). Then for homogeneous elements $d \in D,\ a, a' \in A$, the identity $\mathcal{J}(d, f\otimes a, g\otimes a') = 0$ gives the condition $[d, \langle a, a'\rangle] = \langle da, a'\rangle + (-1)^{\bar d \bar a}\langle a, da'\rangle$. From $\mathcal{J}(f \otimes a, g \otimes a', h \otimes a'') = 0$ with homogeneous $a, a', a'' \in A$ and with $f,\, g,\, h \in \mathfrak{g}$ as in assumption (ii), we determine that $\langle A, A\rangle$ is contained in the kernel of Φ. Finally, $\mathcal{J}(f \otimes a_1, g \otimes a_2, h \otimes a_3) = 0$ for a_1, a_2, a_3 homogeneous and $f, g, h \in \mathfrak{g}$ as in assumption (iii) gives $0 = \sum_{\circlearrowleft} (-1)^{\bar a_1 \bar a_3}\langle a_1a_2, a_3\rangle = 0$ and $(a_2a_3)a_1 = (-1)^{\bar a_2(\bar a_3+\bar a_1)}(a_3a_1)a_2$. By supercommutativity, this is the same as $(a_2a_3)a_1 = a_2(a_3a_1)$, and hence the associativity of A follows.

The converse is a simple computation. $\square$

§5. The Main Theorem

In order to apply Lemma 4.1 to the Δ-graded Lie superalgebras considered here, it has to be checked that both $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$ and $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes \mathfrak{g}, \mathbb{F})$ are one-dimensional, $\mathfrak{g}$ being a split simple classical Lie superalgebra of type $\mathrm{C}(n)$ $(n \geq 3)$, $\mathrm{D}(m, n)$ $(m \geq 2,\, n \geq 1), \mathrm{D}(2, 1; \alpha)$ $(\alpha \in \mathbb{F} \setminus \{0, -1\}),$ $\mathrm{F}(4)$, or $\mathrm{G}(3)$. The existence of a nondegenerate even supersymmetric bilinear form on $\mathfrak{g}$ and the fact that $\mathfrak{g}$ is central simple over $\mathbb{F}$ immediately imply the assertion for $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes \mathfrak{g}, \mathbb{F})$.

Lemma 5.1. *Let $\mathfrak{g}$ be a split simple classical Lie superalgebra of type* $\mathrm{C}(n)$ $(n \geq 3)$, $\mathrm{D}(m, n)$ $(m \geq 2,\, n \geq 1), \mathrm{D}(2, 1; \alpha)$ $(\alpha \in \mathbb{F} \setminus \{0, -1\})$, $\mathrm{F}(4)$, *or* $\mathrm{G}(3)$. *Then* $\dim \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g}) = 1$.

Proof. Assume first that $\mathfrak{g}$ is of type $\mathrm{C}(n)$ $(n \geq 3)$ and consider the $\mathbb{Z}$-gradation used in Section 2, $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, with $\mathfrak{g}_{\bar 0} = \mathfrak{g}_0$ and $\mathfrak{g}_{\bar 1} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$. Then

$\mathfrak{g}_0 = \mathbb{F}c \oplus \mathfrak{sp}_{2r}$, where $c = E_{-1,-1} - E_{0,0}$ as in Section 3, which is central in $\mathfrak{g}_0$, and $r = n-1$. The spaces $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are isomorphic, as $\mathfrak{sp}_{2r}$-modules, to the natural $2r$-dimensional irreducible module for $\mathfrak{sp}_{2r}$, while c acts as the identity on $\mathfrak{g}_1$ and as minus the identity on $\mathfrak{g}_{-1}$. Once the Cartan subalgebra $\mathfrak{h} = \mathbb{F}c \oplus \mathfrak{h}'$ of $\mathfrak{g}_0$ and a system of simple roots are chosen as in (2.19) and (2.20), we may take a highest weight vector $v \in \mathfrak{g}_1$ and a lowest weight vector $w \in \mathfrak{g}_{-1}$ (as $\mathfrak{g}_0$-modules). Then $v \otimes w$ generates $\mathfrak{g} \otimes \mathfrak{g}$ as a $\mathfrak{g}$-module (one gets easily that $\mathfrak{g}_1 \otimes w$ is contained in the $\mathfrak{g}_0$-module generated by $v \otimes w$, and hence that $\mathfrak{g} \otimes w$ is contained in the $\mathfrak{g}$-module generated by $v \otimes w$. But $\mathfrak{g} \otimes w$ generates $\mathfrak{g} \otimes \mathfrak{g}$ as a $\mathfrak{g}$-module). Thus, any $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$ is determined by $\varphi(v \otimes w)$, which belongs to $\mathfrak{h} = \mathbb{F}c \oplus \mathfrak{h}'$ because $v \otimes w$ has weight 0. In particular, φ restricts to a $\mathfrak{g}_0$-module homomorphism $\mathfrak{g}_1 \otimes \mathfrak{g}_{-1} \to \mathfrak{g}_0$. Since $\mathrm{Hom}_{\mathfrak{sp}_{2r}}(\mathfrak{g}_1 \otimes \mathfrak{g}_{-1}, \mathfrak{sp}_{2r})$ has dimension 1 (as $\mathfrak{sp}_{2r}$-modules, this is $\mathrm{Hom}_{\mathfrak{sp}_{2r}}\big(V(\omega_1) \otimes V(\omega_1), V(2\omega_1)\big)$, where ω_1 is the first fundamental dominant weight for $\mathfrak{sp}_{2r}$), it follows that there is $0 \neq h \in \mathfrak{h}'$ such that $\varphi(v \otimes w) \in \mathbb{F}c \oplus \mathbb{F}h$ for any $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$ and $\dim \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g}) \leq 2$. If this dimension were 2, there would exist a $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$ with $\varphi(v \otimes w) = c$ and, therefore, $\varphi(\mathfrak{g}_1 \otimes \mathfrak{g}_{-1}) = \mathbb{F}c$. Then, for any $x \in \mathfrak{g}_1$, $\varphi(\mathfrak{g}_1 \otimes [x, \mathfrak{g}_{-1}]) \subseteq \mathbb{F}[c, x] = \mathbb{F}x$. It is not difficult to find linearly independent elements $x, y \in \mathfrak{g}_1$ such that both $[x, \mathfrak{g}_{-1}]$ and $[y, \mathfrak{g}_{-1}]$ are not contained in $\mathfrak{sp}_{2r}$, and there is a nonzero $z \in [x, \mathfrak{g}_{-1}] \cap [y, \mathfrak{g}_{-1}] \cap \mathfrak{sp}_{2r}$. Then $\varphi(\mathfrak{g}_1 \otimes z) \subseteq \mathbb{F}x \cap \mathbb{F}y = 0$, which implies $\varphi(\mathfrak{g}_1 \otimes \mathfrak{sp}_{2r}) = 0$, since $\mathfrak{sp}_{2r}$ is simple and hence generated by z as a $\mathfrak{g}_0$-module. But then $\varphi(\mathfrak{g}_1 \otimes \mathfrak{g}_0) = \varphi(\mathfrak{g}_1 \otimes c) = \varphi(\mathfrak{g}_1 \otimes [x, \mathfrak{g}_{-1}]) \subseteq \mathbb{F}x$, and also $\varphi(\mathfrak{g}_1 \otimes \mathfrak{g}_0) \subseteq \mathbb{F}y$. Therefore, $\varphi(\mathfrak{g}_1 \otimes \mathfrak{g}_0) = 0$. Since φ is ad_c-invariant, $\varphi(\mathfrak{g}_1 \otimes \mathfrak{g}_1) = 0$ too. In the same way we prove that $\varphi(\mathfrak{g}_0 \otimes \mathfrak{g}_{-1}) = \varphi(\mathfrak{g}_{-1} \otimes \mathfrak{g}_{-1}) = 0$. Finally, $\varphi(\mathfrak{g}_0 \otimes \mathfrak{g}_1) = \varphi([\mathfrak{g}_1, \mathfrak{g}_{-1}] \otimes \mathfrak{g}_1) \subseteq [\mathfrak{g}_{-1}, \varphi(\mathfrak{g}_1 \otimes \mathfrak{g}_1)] + \varphi(\mathfrak{g}_1 \otimes \mathfrak{g}_0) = 0$ and also $\varphi(\mathfrak{g}_{-1} \otimes \mathfrak{g}_0) = 0$. Therefore $\varphi(\mathfrak{g} \otimes \mathfrak{g}) \subseteq \mathfrak{g}_0$, but $0 \neq \varphi(\mathfrak{g} \otimes \mathfrak{g})$ is an ideal of $\mathfrak{g}$, a contradiction.

Assume now that $\mathfrak{g}$ is of type $\mathrm{D}(m,n)$ $(m \geq 2, n \geq 1)$, $\mathrm{D}(2,1;\alpha)$ $(\alpha \in \mathbb{F} \setminus \{0,-1\})$, $\mathrm{F}(4)$, or $\mathrm{G}(3)$. Then $\mathfrak{g}$ has a $\mathbb{Z}$-gradation [K1, §2] $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$, with $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_2$ and $\mathfrak{g}_{\bar{1}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_1$. The spaces $\mathfrak{g}_2$ and $\mathfrak{g}_{-2}$ are irreducible contragredient $\mathfrak{g}_0$-modules as are $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$; $\mathfrak{g}_0 = \mathbb{F}c \oplus [\mathfrak{g}_0, \mathfrak{g}_0]$, where $[\mathfrak{g}_0, \mathfrak{g}_0]$ is a semisimple Lie algebra; and $[c, x_i] = i x_i$ for any $x_i \in \mathfrak{g}_i$, $i = \pm 2, \pm 1, 0$. As before we fix a Cartan subalgebra $\mathfrak{h} = \mathbb{F}c \oplus \mathfrak{h}'$ of $\mathfrak{g}_0$ and take a highest weight vector $v \in \mathfrak{g}_2$ and a lowest weight vector $w \in \mathfrak{g}_{-2}$. Then $v \otimes w$ generates $\mathfrak{g} \otimes \mathfrak{g}$ as a $\mathfrak{g}$-module, and any $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$ is determined by $\varphi(v \otimes w)$, which belongs to $\mathfrak{h}$ (by ad_c-invariance, φ must respect the $\mathbb{Z}$-gradation).

For types $\mathrm{D}(2,1;\alpha)$ $(\alpha \in \mathbb{F} \setminus \{0,-1\})$, $\mathrm{F}(4)$, or $\mathrm{G}(3)$, $\mathfrak{g}_{\pm 2}$ is one-dimensional and annihilated by $[\mathfrak{g}_0, \mathfrak{g}_0]$. Hence $\varphi(v \otimes w) \in \mathbb{F}c$ and, therefore, $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g})$ is one-dimensional. For type $\mathrm{D}(m,n)$ $(m \geq 2, n \geq 1)$, $[\mathfrak{g}_0, \mathfrak{g}_0] = \mathfrak{o}_{2m} \oplus \mathfrak{sl}_n$ and $\mathfrak{g}_2$ and $\mathfrak{g}_{-2}$ are annihilated by $\mathfrak{o}_{2m}$. The argument in [BE1, Proof of (3.5)] applies here to give the result. □

Theorem 5.2. *Assume L is a Δ-graded Lie superalgebra with grading subalgebra $\mathfrak{g}$ corresponding to a root system Δ of type* $\mathrm{C}(n)$ $(n \geq 3)$*,* $\mathrm{D}(m,n)$ $(m \geq 2$*,* $n \geq 1)$*,* $\mathrm{D}(2,1;\alpha)$ $(\alpha \in \mathbb{F} \setminus \{0,-1\})$*,* $\mathrm{F}(4)$*, or* $\mathrm{G}(3)$*. Then there exist a unital supercommutative associative $\mathbb{F}$-superalgebra A and an $\mathbb{F}$-superspace D such that $L \cong (\mathfrak{g} \otimes A) \oplus D$. Multiplication in L is given by*

$$[f\otimes a, g\otimes a'] = (-1)^{\bar a\bar g}\Big([f,g]\otimes aa' + \kappa(f,g)\langle a,a'\rangle\Big)$$
$$[d,L]=0$$

for all $f,g\in\mathfrak g$, $a,a'\in A$, $d\in D$, where $\kappa(f,g)$ is a fixed even nondegenerate supersymmetric bilinear form on $\mathfrak g$, and $\langle\ ,\ \rangle : A\times A\to D$ is $\mathbb F$-bilinear and superskewsymmetric and satisfies the two-cocycle condition, $\sum_{\circlearrowleft}(-1)^{\bar a_1\bar a_3}\langle a_1a_2,a_3\rangle = 0$.

Proof. The results of Sections 2 and 3 show that every such Δ-graded Lie superalgebra L is a direct sum of adjoint and trivial modules. Most of the conclusions of the theorem will be immediate consequences of Lemma 4.1, once we verify that the hypotheses in (1)' and (2)' of that lemma are satisfied. The fact $\dim\mathrm{Hom}_{\mathfrak g}(\mathfrak g\otimes\mathfrak g,\mathbb F) = 1 = \dim\mathrm{Hom}_{\mathfrak g}(\mathfrak g\otimes\mathfrak g,\mathfrak g)$ comes from Lemma 5.1 and the paragraph preceding it. When $\mathfrak g_{\bar 0}$ is a reductive Lie algebra of rank at least 2 (which happens in all our cases), conditions (i)–(iii) of (2)' are always satisfied. Indeed, assume we have a root space decomposition of $\mathfrak g_{\bar 0}$ relative to the Cartan subalgebra $\mathfrak h$. For (i) take f in a root space (say of root α) and g in the root space corresponding to the root $-\alpha$; while for (ii) and (iii) choose f,g as before. Let $h\in\mathfrak h$ be such that $\alpha(h)=0$ for (ii); and for (iii), take $h\in\mathfrak h$ with $\alpha(h)\neq 0$.

The only point left is the proof of the centrality of D. Condition (ii) of Definition 1.2 implies that $L_0=\sum_{\mu\in\Delta}[L_\mu,L_{-\mu}]$. This forces $D=\langle A\mid A\rangle$, which by Lemma 4.1 is contained in $\ker\Phi$. Therefore $D=\langle A\mid A\rangle$ is abelian and centralizes $\mathfrak g\otimes A$, hence it is central. $\square$

Recall that a *central extension* of a Lie superalgebra L is a pair $(\widetilde L,\pi)$ consisting of a Lie superalgebra $\widetilde L$ and a surjective Lie superalgebra homomorphism $\pi:\widetilde L\to L$ (preserving the grading) whose kernel lies in the center of $\widetilde L$. If $\widetilde L$ is perfect ($\widetilde L = [\widetilde L,\widetilde L]$), then $\widetilde L$ is said to be a *cover* or *covering* of L. Any perfect Lie superalgebra L has a unique (up to isomorphism) universal covering superalgebra $(\widehat L,\widehat\pi)$ which is also perfect, called the *universal central extension* of L. From Theorem 5.2 we can draw the conclusion that our Δ-graded Lie superalgebras are coverings:

Corollary 5.3. *A Δ-graded Lie superalgebra with grading subalgebra $\mathfrak g$ corresponding to a root system Δ of type* $\mathrm C(n)$ $(n\geq 3)$, $\mathrm D(m,n)$ $(m\geq 2,\ n\geq 1)$, $\mathrm D(2,1;\alpha)$ $(\alpha\in\mathbb F\setminus\{0,-1\})$, $\mathrm F(4)$, *or* $\mathrm G(3)$ *is a covering of a Lie superalgebra $\mathfrak g\otimes A$, where A is a unital supercommutative associative superalgebra.*

Suppose now that A is a unital supercommutative associative superalgebra. Set $\{A\mid A\} = (A\otimes A)/I$, where I is the subspace spanned by the elements $a_1\otimes a_2 + (-1)^{\bar a_1\bar a_2}a_2\otimes a_1$ and $\sum_{\circlearrowleft}(-1)^{\bar a_1\bar a_3}a_1a_2\otimes a_3$ ($a_i\in A_{\bar 0}\cup A_{\bar 1}$, $i=1,2,3$). As a shorthand we write $\{a\mid a'\} = a\otimes a' + I$. Then it follows from Theorem 5.2 that the universal central extension of the Lie superalgebra $L=\mathfrak g\otimes A$ is

$$\widehat L = \big(\mathfrak g\otimes A\big)\oplus\{A\mid A\} \tag{5.4}$$

with $\{A\mid A\}$ central and with

$$[f\otimes a, g\otimes a'] = (-1)^{\bar a\bar g}\Big([f,g]\otimes aa' + \kappa(f,g)\{a\mid a'\}\Big) \tag{5.5}$$

for all $f, g \in \mathfrak{g}$ and $a, a' \in A$. In the special case that A is a commutative associative algebra, this result appears in [IK].

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Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706
E-mail address: `benkart@math.wisc.edu`

Departamento de Matemáticas, Universidad de Zaragoza, 50009 Zaragoza, Spain
E-mail address: `elduque@posta.unizar.es`