Homotopes of Symmetric Spaces
I. Construction by Algebras with Two Involutions

Wolfgang Bertram
IECN Université Nancy I, France
e-mail: bertram@iecn.u-nancy.fr
Pierre Bieliavsky
Université Catholique de Louvain, Belgium.
e-mail: bieliavsky@math.ucl.ac.be

Abstract. We investigate a special kind of contraction of symmetric spaces (respectively, of Lie triple systems), called *homotopy.* In this first part of a series of two papers we construct such contractions for classical symmetric spaces in an elementary way by using *associative algebras with several involutions.* This construction shows a remarkable duality between the underlying "space" and the "deformation parameter".

Subject classification (2010). 16W10, 17B60, 17C37, 53C35, 32M15

Key words. associative algebra, contraction, involution, homotope (isotope), Lie algebra, Lie triple system, symmetric space, extrinsic symmetric space

Introduction

A glance at the classification of simple symmetric spaces ([Berger57]) shows that most of the classical symmetric spaces can be directly defined in terms of the classical matrix algebras $\mathbb{A} = M(n,n;\mathbb{F})$, $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, together with one, two or at most three involutions. In this paper we revisit these constructions from an algebraic viewpoint, working with *Lie triple systems* (which are infinitesimal versions of symmetric spaces), and we show that they define not only the discrete families of simple spaces, but more generally "continuous families" of symmetric spaces, including a great variety of non-reductive spaces; these continuous families can be considered as a special kind *contractions*, called *homotopes of symmetric spaces* (introduced in [Be08]). We intend to use such contractions in further work to generalize quantization procedures from [BDS09].

The basic idea of the construction is quite simple: in the setting of an associative algebra $\mathbb{A}$, "homotopy" amounts to the observation that, for any $A \in \mathbb{A}$, the product $(X, Y) \mapsto XAY$ on $\mathbb{A}$ is again associative, hence $[X, Y]_A := XAY - YAX$ is a Lie bracket, and

$$[X, Y, Z]_A := [[X, Y]_A, Z]_A = (XAYAZ + ZAYAX) - (YAXAZ + ZAXAY) \tag{1}$$

is a Lie triple product on $\mathbb{A}$. For $A = 1$ (unit element), this is the "standard" or "general linear" Lie triple product, and for $A = 0$ it is the "flat" Lie triple product on $\mathbb{A}$; thus we may say that the family of Lie triple products indexed by $A \in \mathbb{A}$ is a contraction of the "general linear" Lie triple product. Now assume that the associative algebra $\mathbb{A}$ carries several commuting involutions $\tau_1, \ldots, \tau_k$; for most of the constructions $k = 1$ or 2 will be sufficient. Let $\mathfrak{m}$ be any of the 2^k joint eigenspaces of these involutions. The key observation is now: if A belongs to some joint eigenspace, then $\mathfrak{m}$ is stable under $[X, Y, Z]_A$, hence is a Lie triple system (Lemma 3.3). This means that we have 4^k families of Lie triple systems, which can be organized in a table forming a "$2^k \times 2^k$-matrix", with rows containing homotopes living on the same underlying space $\mathfrak{m}$ and columns containing homotopes parametrized by the same parameter space. Such kind of "duality" between space and deformation parameter is a special feature of the classical spaces governed by *associative* algebras considered here; it is not present in the general (Jordan theoretic) formulation to be discussed in Part II of this work.

We give several examples of such "matrix-tables" (Theorems 3.7, 3.8, 3.9, 3.10); all of them reflect interesting features of families of classical symmetric spaces, among them, e.g., the *Siegel upper half plane* $\mathrm{Sp}_n(\mathbb{R})/\mathrm{U}(n)$ (Theorem 3.8) and its compact dual $\mathrm{Sp}(n)/\mathrm{U}(n)$ (Theorem 3.9). The duality between lines and columns of these tables makes it now easy to calculate an *algebra imbedding* of the Lie triple system $\mathfrak{m}$, that is, a Lie algebra $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a symmetric Lie algebra decomposition: if $\mathfrak{m}$ belongs to the antidiagonal of the table, then $\mathfrak{m}$ is already a Lie algebra; else we may choose for $\mathfrak{h}$ the space from the antidiagonal that belongs to the same column as $\mathfrak{m}$ (see Theorem 3.4 and its proof).

For a fixed joint eigenspace $\mathfrak{m}$, the Lie triple systems $\mathfrak{m}$ with Lie triple product $[X,Y,Z]_A$ can be interpreted as curvature tensors of symmetric spaces $M = G/H$ depending on A and all having the same tangent space at the origin. In Section 4, we focus on the real, finite-dimensional case and give a list of all families of symmetric spaces obtained that way, organized according to the point of view of "deformations" and "contractions": for each underlying space $\mathfrak{m}$ we give a list of Lie triple products on $\mathfrak{m}$ that are homotopes of each other (Theorem 4.2).

This work is organized as follows: basic facts on Lie triple systems and classical Lie algebras are recalled in Chapters 1 and 2; in Chapter 3 the "two-involution-construction" is explained, and the most important examples are woked out. Chapter 4 contains the systematic list of contractions of symmetric spaces thus obtained. In Part II of this work ([BeBi]) we explain a more general construction of homotopes, which uses basic ideas of *Jordan theory* (as we hope to convince the reader, the concept of homotopy is useful in the associative theory, but its rôle in the non-associative theory is certainly even more important): we define the *structure variety*, which is the natural parameter space for contractions, and we show that the list given here in Chapter 4 is, essentially, a complete description of the corresponding structure varieties. While proving this, we will also obtain results describing in more detail the structure of the contracted spaces: a non-reductive homotope has a *bundle structure*, with flat fibers and a reductive base, and we will analyze some interesting low-dimensional examples of such fibered symmetric spaces.

Acknowledgements. W.B. thanks Université Catholique de Louvain for hospitality in 2010 when part of this work was carried out. P.B. thanks Université Henri Poincaré-Nancy I for hospitality and the Belgian Scientific Policy (BELSPO) for its support through the IAP 'NOSY' to which he is affiliated at the Université Catholique de Louvain.

Notation. Throughout this paper $\mathbb{K}$ is a commutative base ring in which 2 is invertible.

1 Lie triple systems

A *Lie triple system (LTS)* is a $\mathbb{K}$-module $\mathfrak{q}$ together with a trilinear map

$$\mathfrak{q}^3 \to \mathfrak{q}, \quad (X,Y,Z) \mapsto [X,Y,Z] =: R(X,Y)Z$$

satisfying, for all $X,Y,Z,U,V \in \mathfrak{q}$,

(LT1) $[X,Y,Z] = -[Y,X,Z]$

(LT2) $[X,Y,Z] + [Y,Z,X] + [Z,X,Y] = 0$

(LT3) the endomorphism $D := R(U,V)$ is a derivation of the trilinear product $[X,Y,Z]$.

Every Lie algebra $\mathfrak{g}$ with $[X,Y,Z] := [[X,Y],Z]$ is a LTS, and if σ is an automorphism of $\mathfrak{g}$ of order 2, then the -1-eigenspace $\mathfrak{q}$ of σ is stable under this trilinear product and hence is a LTS. Every LTS $\mathfrak{q}$ is obtained in this way: we may take for $\mathfrak{g}$ the *standard imbedding* $\mathfrak{q} \oplus [\mathfrak{q},\mathfrak{q}] \subset \mathfrak{q} \oplus \mathfrak{Der}(\mathfrak{q})$ (see [Lo69]). The pair $(\mathfrak{g},\sigma)$ is called a *symmetric pair*. If $\mathfrak{h} = \mathfrak{g}^\sigma$ is the fixed point algebra of σ, we will sometimes, with some abuse of notation, denote the symmetric pair also by $(\mathfrak{g},\mathfrak{h})$. This notation is motivated (in the real finite dimensional case) by the usual description of the *associated symmetric space* as a homogeneous space $M = G/H$, where G is a Lie group with involution σ and H an open subgroup of the fixed point group G^σ. The case of a Lie group H considered as symmetric space $H \times H/diag(H \times H)$ will be called a *group case*; it corresponds to the case of a Lie algebra, considered as a LTS.

c-duality. It is clear that, if $(\mathfrak{q}, R)$ is a LTS, then all multiples $(\mathfrak{q}, \lambda R)$ for $\lambda \in \mathbb{K}$ are again LTS. In particular, $(\mathfrak{q}, -R)$ is again a LTS, called the *c-dual Lie triple system*, where the letter c refers to "compact" or "Cartan": indeed, in the real finite dimensional case, R is of compact type if and only if $-R$ is of non-compact type. Note that, if λ is a square in $\mathbb{K}^\times$, then $\lambda \mathrm{id}_{\mathfrak{q}}$ is an isomorphism between R and $\lambda^2 R$. In particular, for $\mathbb{K} = \mathbb{C}$, the LTS R and $-R$ are always isomorphic to each other. For $\mathbb{K} = \mathbb{R}$, if $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ is the standard imbedding of R, then $\mathfrak{h} \oplus i\mathfrak{q}$ (subalgebra of $\mathfrak{g}_{\mathbb{C}}$) is the standard imbedding of $-R$. Using this, it is seen, for example, that the c-dual of a group case H is a symmetric space of the form $H_{\mathbb{C}}/H$, where $H_{\mathbb{C}}$ is a complexification of H.

2 Classical Lie algebras

We call "classical" Lie algebras that are defined by means of an associative unital $\mathbb{K}$-algebra $\mathbb{A}$ with some involution τ (antiautomorphism of order 2); we often use the notation a^* for $\tau(a)$ and denote the eigenspace decomposition of $*$ by $\mathbb{A} = \mathbb{A}^\tau \oplus \mathbb{A}^{-\tau} = \mathrm{Herm}(\mathbb{A}, *) \oplus \mathrm{Aherm}(\mathbb{A}, *)$. We are going to define families of classical Lie algebras associated to these data, parametrized by certain elements $A \in \mathbb{A}$. The three main types of classical Lie algebras are given by

Lemma 2.1 *The following data define Lie algebras:*

(1) general linear: $\mathfrak{g}_A := \mathfrak{g}_{\mathbb{A},A} := \mathbb{A}$ *with* $[X,Y]_A := XAY - YAX$, *for any* $A \in \mathbb{A}$,

(2) unitary / orthogonal: $\mathfrak{u}_A := \mathfrak{u}_{\mathbb{A},A,\tau} := \mathrm{Aherm}(\mathbb{A}, *)$ *with* $[X,Y]_A$, *for any* $A \in \mathrm{Herm}(A, *)$,

(3) (half) symplectic: $\mathfrak{sp}_A := \mathfrak{sp}_{\mathbb{A},A,\tau} := \mathrm{Herm}(\mathbb{A}, *)$ *with* $[X,Y]_A$, *for any* $A \in \mathrm{Aherm}(A, *)$.

Proof. (1) follows from the fact that $(X,Y) \mapsto XAY$ is an associative product, and (2) and (3) from the fact that $*$ is an antiautomorphism (resp. automorphism) of the bracket $[X,Y]_A$. ■

The "classical Lie algebras" are obtained by taking $\mathbb{A} = M(n,n;\mathbb{F})$, the matrix algebra over $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, with involution $X^* = \delta(X)^t$ (transposed matrix of $\delta(X)$, where $\delta : \mathbb{F} \to \mathbb{F}$ is an involution of the base field $\mathbb{F}$; the various choices for δ will be specified below). For the moment, $(\mathbb{F}, \delta)$ may be any unital ring with involution. We fix notation and terminology as follows (cf. [BeKi09]):

family name	label and space	parameter space	Lie bracket
general linear (square)	$\mathfrak{gl}_n(A;\mathbb{F}) := M(n,n;\mathbb{F})$	$A \in M(n,n;\mathbb{F})$	$[X,Y]_A$
$(\mathbb{F},\delta)$-unitary	$\mathfrak{u}_n(A;\mathbb{F},\delta) := \mathrm{Aherm}(n;\mathbb{F},\delta)$	$A \in \mathrm{Herm}(n;\mathbb{F},\delta)$	$[X,Y]_A$
$(\mathbb{F},\delta)$-symplectic	$\mathfrak{sp}_{n/2}(A;\mathbb{F},\delta) := \mathrm{Herm}(n;\mathbb{F},\delta)$	$A \in \mathrm{Aherm}(n;\mathbb{F},\delta)$	$[X,Y]_A$

Remark. The general linear type can be defined, more generally, for *associative pairs* (see [BeKi09]); in the table below this type appears as *rectangular matrices* (second line). For $p \neq q$ these algebras are never reductive. Similarly, for odd n, the symplectic type is never reductive, and we then prefer to call it *half-symplectic.*

Now we specify the involution of the base field or ring $\mathbb{F}$. In the following table, $\mathbb{F}$ is one of the skew-fields $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{K}$ one of the fields $\mathbb{R}$, $\mathbb{C}$. Concerning involutions: for $\mathbb{F} = \mathbb{C}$, we always use usual complex conjugation, and for $\mathbb{K} = \mathbb{H}$, if nothing else is specified, we use "usual" conjugation $\lambda \mapsto \overline{\lambda}$ (minus one in the imaginary part $\mathtt{Im}\mathbb{H}$ and one on the center $\mathbb{R} \subset \mathbb{H}$). If we consider $\mathbb{H}$ with its "split" involution $\lambda \mapsto \widetilde{\lambda} := j\overline{\lambda}j^{-1}$, then we write $\widetilde{\mathbb{H}}$. For instance, $\mathrm{Herm}(n;\widetilde{\mathbb{H}})$ is the space of quaternionic matrices such that $\widetilde{X} = X^t$, hence $\mathfrak{u}_n(1;\widetilde{\mathbb{H}})$ is the Lie algebra in the literature often denoted by $\mathfrak{so}^*_{2n}$.

family name	label and space	parameter space	Lie bracket
general linear (square)	$\mathfrak{gl}_n(A;\mathbb{F}) := M(n,n;\mathbb{F})$	$A \in M(n,n;\mathbb{F})$	$[X,Y]_A$
general linear (rectan.)	$\mathfrak{gl}_{p,q}(A;\mathbb{F}) := M(p,q;\mathbb{F})$	$A \in M(q,p;\mathbb{F})$	$[X,Y]_A$
orthogonal	$\mathfrak{o}_n(A;\mathbb{K}) := \mathrm{Asym}(n;\mathbb{K})$	$A \in \mathrm{Sym}(n;\mathbb{K})$	$[X,Y]_A$
[half-] symplectic	$\mathfrak{sp}_{n/2}(A;\mathbb{K}) := \mathrm{Sym}(n;\mathbb{K})$	$A \in \mathrm{Asym}(n;\mathbb{K})$	$[X,Y]_A$
$\mathbb{C}$-unitary	$\mathfrak{u}_n(A;\mathbb{C}) := \mathrm{Aherm}(n;\mathbb{C})$	$A \in \mathrm{Herm}(n;\mathbb{C})$	$[X,Y]_A$
$\mathbb{H}$-unitary	$\mathfrak{u}_n(A;\mathbb{H}) := \mathrm{Aherm}(n;\mathbb{H})$	$A \in \mathrm{Herm}(n;\mathbb{H})$	$[X,Y]_A$
$\mathbb{H}$-unitary split	$\mathfrak{u}_n(A;\widetilde{\mathbb{H}}) := \mathrm{Aherm}(n;\widetilde{\mathbb{H}})$	$A \in \mathrm{Herm}(n;\widetilde{\mathbb{H}})$	$[X,Y]_A$

Classification up to isomorphy of algebras from the preceding table is easy, by using the following general

Lemma 2.2 *Let A, T, S be arbitrary elements of the associative algebra $\mathbb{A}$. Then*

$$\mathfrak{g}_{TAS} \to \mathfrak{g}_A, \quad X \mapsto SXT$$

is a Lie algebra homomorphism. In particular, if $g, h \in \mathbb{A}^\times$, the Lie algebras $\mathfrak{g}_{gAh}$ and $\mathfrak{g}_A$ are isomorphic, and so are $\mathfrak{u}_{gAg^*}$ and $\mathfrak{u}_A$ (resp. $\mathfrak{sp}_{gAg^*}$ and $\mathfrak{sp}_A$).

Proof. $[SXT, SYT]_A = SXTASYT - SYTASXT = S[X,Y]_{TAS}T$ ∎

In the case of matrix algebras over a field, any A is conjugate to a matrix which is *idempotent*, and classification of algebras is reduced the well-known classification of (Hermitian or skew-Hermitian) idempotents. Moreover, if A is idempotent, or more generally if $A^3 = A$, we may apply the lemma with $T = S = A$ to get an algebra endomorphism of $\mathfrak{g}_A$, which leads to the *fibered structure of homotopes* to be studied in more detail in Part II.

3 Classical Lie triple systems

Next we are going to construct families of "classical Lie triple systems" in a similar way as above. The construction will be based on associative algebras with *several commuting involutions*. However, first of all let us review the preceding situation (no, or just one, involution) from the point of view of Lie triple systems.

3.1 Associative algebras with one involution

In an associative algebra, we will use the notation

$$T(X,Y,Z) := XYZ + ZYX\,. \tag{2}$$

Lemma 3.1 *Let $\mathbb{A}$ be an associative algebra, $A \in \mathbb{A}$ and $\alpha(X) := AXA$. Then $\mathbb{A}$ with ternary bracket*

$$[X,Y,Z]_A := (XAYAZ + ZAYAX) - (YAXAZ + ZAXAY) = T(X,\alpha Y,Z) - T(Y,\alpha X,Z)$$

is the Lie triple system belonging to the Lie algebra $(\mathbb{A}, [X,Y]_A)$.

Proof. $[[X,Y]_A, Z]_A = (XAY - YAX)AZ - ZA(XAY - YAX) = T(X,\alpha Y,Z) - T(Y,\alpha X,Z)$. ∎

One may note that $[X,Y,Z]_{rA} = r^2 [X,Y,Z]_A$ for $r \in \mathbb{K}$, whence $[X,Y,Z]_A = [X,Y,Z]_{-A}$, and, if $\mathbb{K} = \mathbb{C}$, the c-dual LTS is obtained in the form $[X,Y,Z]_{iA} = -[X,Y,Z]_A$.

Lemma 3.2 *Assume $\tau(X) := X^*$ is an involution of the associative algebra $\mathbb{A}$. If A belongs to one of the eigenspaces, then both eigenspaces are stable under $[X,Y,Z]_A$. This gives four possibilities to combine choices, leading to the following four families of Lie triple systems:*

(1) *Assume $A \in \mathbb{A}^\tau$. Then $\mathbb{A}^{-\tau}$ is the LTS belonging to the unitary Lie algebra $\mathfrak{u}_A$, and the space $\mathbb{A}^\tau$ with $[X,Y,Z]_A$ is the LTS belongs to the symmetric pair $(\mathfrak{g}_A, \mathfrak{u}_A)$.*

(2) *Assume $A \in \mathbb{A}^{-\tau}$. Then $\mathbb{A}^\tau$ is the LTS belonging to the (half-)symplectic Lie algebra $\mathfrak{sp}_A$, and the space $\mathbb{A}^{-\tau}$ with $[X,Y,Z]_A$ is the LTS belongs to the symmetric pair $(\mathfrak{g}_A, \mathfrak{sp}_A)$.*

We summarize these statements by the following table:

	$A \in \mathbb{A}^\tau$	$A \in \mathbb{A}^{-\tau}$
LTS $\mathbb{A}^\tau$	$(\mathfrak{g}_{\mathbb{A},A}, \mathfrak{u}_{\mathbb{A},A,\tau})$	$\mathfrak{sp}_{A,\tau}$
LTS $\mathbb{A}^{-\tau}$	$\mathfrak{u}_{\mathbb{A},A,\tau}$	$(\mathfrak{g}_{\mathbb{A},A}, \mathfrak{sp}_{\mathbb{A},A,\tau})$

Proof. If $\tau(A) = \mp A$, then τ is either a Lie algebra automorphism or antiautomorphism of $\mathfrak{g}_A$, and hence in both cases it is a LTS-automorphism, and hence both eigenspaces are sub-LTS. Now both claims are immediate consequences of the definition of $\mathfrak{sp}_A$ and $\mathfrak{u}_A$. ∎

Note that the LTS of group cases are found on the *antidiagonal* of the table; this reflects the fact that τ is an *anti*automorphism of $\mathbb{A}$. If one is interested in isomorphism classes of the LTS from the lemma, one may observe that the group $\mathbb{A}^\times$ acts on both eigenspaces by $(a,x) \mapsto ax\tau(a)$, and if A and A' are conjugate under this action, then the LTS indexed by A and A' are isomorphic.

3.2 Associative algebras with two commuting involutions

Now assume that $\mathbb{A}$ carries two commuting involutions τ and $\tilde{\tau}$, and let $\phi := \tau \circ \tilde{\tau}$. This is an automorphism of order 2, and conversely, given an automorphism of order 2 commuting with τ, we recover $\tilde{\tau} = \phi \circ \tau$. For a given pair $(\tau_1, \tau_2) = (\tau, \tilde{\tau})$, will denote joint eigenspaces by a double superscript, for instance

$$\mathbb{A}^{(1,-1)} := \mathbb{A}^{(\tau,-\tilde{\tau})} := \mathbb{A}^{\tau} \cap \mathbb{A}^{-\tilde{\tau}} = \{X \in \mathbb{A} \mid \tau(X) = X, \, \tilde{\tau}(X) = -X\},$$

and so on. We thus have a decomposition

$$\mathbb{A} = \mathbb{A}^{(1,1)} \oplus \mathbb{A}^{(1,-1)} \oplus \mathbb{A}^{(-1,1)} \oplus \mathbb{A}^{(-1,-1)},$$

and $\mathbb{A}^{\phi} = \mathbb{A}^{(1,1)} \oplus \mathbb{A}^{(-1,-1)}$ is an associative algebra, whereas the other spaces are in general not associative algebras. Nevertheless, the space $\mathbb{A}^{-\phi} = \mathbb{A}^{(1,-1)} \oplus \mathbb{A}^{(-1,1)}$ is an *associative triple system*, i.e., closed under the ternary associative product XYZ. In particular, for $A \in \mathbb{A}^{-\phi}$, the space $\mathbb{A}^{-\phi}$ is stable under the associative product $(X,Y) \mapsto XAY$, and hence under the Lie bracket $[X,Y]_A$. We denote this Lie algebra by $\mathfrak{g}_{\mathbb{A}^{-\phi},A}$.

Lemma 3.3 *If A belongs to any one of the joint eigenspaces, then all four joint eigenspaces are stable under the triple bracket $[X,Y,Z]_A$.*

Proof. This follows immediately from Lemma 3.2 applied to τ and to $\tilde{\tau}$. ■

Obviously, the lemma generalizes to the case of k commuting involutions: we then have 2^k choices for joint eigenspaces and 2^k choices for parameter spaces, leading to 4^k families of Lie triple systems. The following theorem shows that in our case ($k = 2$), since the roles of τ and $\tilde{\tau}$ are symmetric, the effective number reduces from 16 to about 10:

Theorem 3.4 *The following holds with respect to the Lie triple bracket $[X,Y,Z]_A$.*

(1) *Assume $A \in \mathbb{A}^{(-\tau,-\tilde{\tau})}$. Then the Lie algebras $\mathfrak{sp}_{A,\tau}$ and $\mathfrak{sp}_{A,\tilde{\tau}}$ are defined, and their intersection $\mathfrak{h} := \mathfrak{sp}_{A,\tau} \cap \mathfrak{sp}_{A,\tilde{\tau}}$ is the symplectic Lie algebra $\mathfrak{sp}_{\mathbb{A}^{\phi},A,\tau}$ belonging to the involution τ restricted to the associative algebra $\mathbb{A}^{\phi}$. With this notation, we have:*

- **(i)** $\mathbb{A}^{(\tau,\tilde{\tau})} = \mathfrak{h}$ *is the LTS belonging to the Lie algebra* $\mathfrak{h}$*;*
- **(ii)** $\mathbb{A}^{(-\tau,-\tilde{\tau})}$ *is the LTS belonging to the symmetric pair* $(\mathfrak{g}_{\mathbb{A}^{\phi},A}, \mathfrak{h})$*;*
- **(iii)** $\mathbb{A}^{(\tau,-\tilde{\tau})}$ *is the LTS belonging to the symmetric pair* $(\mathfrak{sp}_{A,\tau}, \mathfrak{h})$*;*
- **(iv)** $\mathbb{A}^{(-\tau,\tilde{\tau})}$ *is the LTS belonging to the symmetric pair* $(\mathfrak{sp}_{A,\tilde{\tau}}, \mathfrak{h})$*.*

(2) *Assume $A \in \mathbb{A}^{(\tau,\tilde{\tau})}$. Then the Lie algebras $\mathfrak{u}_{A,\tau}$ and $\mathfrak{u}_{A,\tilde{\tau}}$ are defined, and their intersection $\mathfrak{h} := \mathfrak{u}_{A,\tau} \cap \mathfrak{u}_{A,\tilde{\tau}}$ is the unitary Lie algebra $\mathfrak{u}_{\mathbb{A}^{\phi},A,\tau}$ belonging to the involution τ restricted to the associative algebra $\mathbb{A}^{\phi}$. With this notation, we have:*

- **(i)** $\mathbb{A}^{(-\tau,-\tilde{\tau})} = \mathfrak{h}$ *is the LTS belonging to the Lie algebra* $\mathfrak{h}$*;*
- **(ii)** $\mathbb{A}^{(\tau,\tilde{\tau})}$ *is the LTS belonging to the symmetric pair* $(\mathfrak{g}_{\mathbb{A}^{\phi},A}, \mathfrak{h})$*;*
- **(iii)** $\mathbb{A}^{(-\tau,\tilde{\tau})}$ *is the LTS belonging to the symmetric pair* $(\mathfrak{u}_{A,\tau}, \mathfrak{h})$*;*
- **(iv)** $\mathbb{A}^{(\tau,-\tilde{\tau})}$ *is the LTS belonging to the symmetric pair* $(\mathfrak{u}_{A,\tilde{\tau}}, \mathfrak{h})$*.*

(3) *Assume $A \in \mathbb{A}^{(\tau,-\tilde{\tau})}$. Then $\mathfrak{h} := \mathfrak{u}_{A,\tau} \cap \mathfrak{sp}_{A,\tilde{\tau}} = \mathbb{A}^{(-\tau,\tilde{\tau})}$ is a Lie algebra, and*

- **(i)** $\mathbb{A}^{(-\tau,\tilde{\tau})}$ *is the LTS belonging to the Lie algebra* $\mathfrak{h}$*;*
- **(ii)** $\mathbb{A}^{(\tau,-\tilde{\tau})}$ *is the LTS belonging to the symmetric pair* $(\mathfrak{g}_{\mathbb{A}^{-\phi},A}, \mathfrak{h})$*;*
- **(iii)** $\mathbb{A}^{(-\tau,-\tilde{\tau})}$ *is the LTS belonging to the symmetric pair* $(\mathfrak{u}_{A,\tau}, \mathfrak{h})$*;*
- **(iv)** $\mathbb{A}^{(\tau,\tilde{\tau})}$ *is the LTS belonging to the symmetric pair* $(\mathfrak{sp}_{A,\tilde{\tau}}, \mathfrak{h})$*.*

We summarize these statements by the following table:

	$A \in \mathbb{A}^{(1,1)}$	$A \in \mathbb{A}^{(-1,1)}$	$A \in \mathbb{A}^{(1,-1)}$	$A \in \mathbb{A}^{(-1,-1)}$
LTS $\mathbb{A}^{(1,1)}$	$(\mathfrak{g}_{\mathbb{A}^\phi,A}, \mathfrak{u}_{\mathbb{A}^\phi,A,\tau})$	$(\mathfrak{sp}_{A,\tau}, \mathfrak{sp}_{A,\tau} \cap \mathfrak{u}_{A,\tilde\tau})$	$(\mathfrak{sp}_{A,\tilde\tau}, \mathfrak{u}_{A,\tau} \cap \mathfrak{sp}_{A,\tilde\tau})$	$\mathfrak{sp}_{\mathbb{A}^\phi,A,\tau}$
LTS $\mathbb{A}^{(-1,1)}$	$(\mathfrak{u}_{\mathbb{A},A,\tau}, \mathfrak{u}_{\mathbb{A}^\phi,A,\tau})$	$(\mathfrak{g}_{\mathbb{A}^{-\phi},A}, \mathfrak{sp}_{A,\tau} \cap \mathfrak{u}_{A,\tilde\tau})$	$\mathfrak{u}_{A,\tau} \cap \mathfrak{sp}_{A,\tilde\tau}$	$(\mathfrak{sp}_{\mathbb{A},A,\tau}, \mathfrak{sp}_{\mathbb{A}^\phi,A,\tau})$
LTS $\mathbb{A}^{(1,-1)}$	$(\mathfrak{u}_{\mathbb{A},A,\tilde\tau}, \mathfrak{u}_{\mathbb{A}^\phi,A,\tau})$	$\mathfrak{sp}_{A,\tau} \cap \mathfrak{u}_{A,\tilde\tau}$	$(\mathfrak{g}_{\mathbb{A}^{-\phi},A}, \mathfrak{u}_{A,\tau} \cap \mathfrak{sp}_{A,\tilde\tau})$	$(\mathfrak{sp}_{\mathbb{A},A,\tilde\tau}, \mathfrak{sp}_{\mathbb{A}^\phi,A,\tau})$
LTS $\mathbb{A}^{(-1,-1)}$	$\mathfrak{u}_{\mathbb{A}^\phi,A,\tau}$	$(\mathfrak{u}_{A,\tilde\tau}, \mathfrak{sp}_{A,\tau} \cap \mathfrak{u}_{A,\tilde\tau})$	$(\mathfrak{u}_{A,\tau}, \mathfrak{u}_{A,\tau} \cap \mathfrak{sp}_{A,\tilde\tau})$	$(\mathfrak{g}_{\mathbb{A}^\phi,A}, \mathfrak{sp}_{\mathbb{A}^\phi,A,\tau})$

Proof. Let us explain the general pattern for k commuting involutions $\tau_1, \ldots, \tau_k$. Given a vector of eigenvalues $\mathbf{s} \in \{\pm 1\}^k$, let $\mathbb{A}^{\mathbf{s}} = \cap_{i=1}^k \mathbb{A}^{s_i \tau_i}$ be the corresponding joint eigenspace. Fix a vector of eigenvalues $\mathbf{t}$ and assume that $A \in \mathbb{A}^{\mathbf{t}}$. Then the joint eigenspace $\mathbb{A}^{-\mathbf{t}}$ is a Lie algebra with respect to the bracket $[X,Y]_A$ (each $\mathbb{A}^{-t_i\tau_i}$ is a Lie algebra for this bracket, according to Lemma 3.2, and $\mathbb{A}^{-\mathbf{t}}$ is the intersection of these algebras), and hence $\mathfrak{h} = \mathbb{A}^{-\mathbf{t}}$ with $[X,Y,Z]_A$ is the LTS belonging to this Lie algebra. This explains item (i) in each case (the antidiagonal of the table). The other three items correspond to the symmetric pair given by the direct sum of $\mathfrak{h}$ with one of the three joint eigenspaces other than the one from (i): if $\mathbf{s}$ is different from $-\mathbf{t}$, then $\mathbb{A}^{\mathbf{s}}$ is not a Lie algebra with respect to $[X,Y]_A$, but $\mathbb{A}^{\mathbf{s}} \oplus \mathbb{A}^{-\mathbf{t}}$ is (this follows since $[\mathbb{A}^{s_i\tau_i}, \mathbb{A}^{s_i\tau_i}]_A \subset \mathbb{A}^{-s_i\tau_i}$ if $s_i = t_i$ and $[\mathbb{A}^{s_i\tau_i}, \mathbb{A}^{s_i\tau_i}]_A \subset \mathbb{A}^{s_i\tau_i}$ if $s_i = -t_i$). Moreover, for all i with $s_i = t_i$, the restriction of τ_i to the Lie algebra $\mathbb{A}^{\mathbf{s}} \oplus \mathbb{A}^{-\mathbf{t}}$ leads to the same Lie algebra automorphism with fixed algebra $\mathfrak{h} = \mathbb{A}^{-\mathbf{t}}$, and hence the LTS $\mathbb{A}^{\mathbf{s}}$ belongs to the symmetric Lie algebra $(\mathbb{A}^{\mathbf{s}} \oplus \mathbb{A}^{-\mathbf{t}}, \mathfrak{h})$.

For $k = 2$, the Lie algebras $\mathbb{A}^{\mathbf{s}} \oplus \mathbb{A}^{-\mathbf{t}}$ have explicit descriptions as follows: let $\mathbf{s} = \mathbf{t}$, that is, A belongs to the underlying space of the LTS in question; then for $\mathbf{t} = (-1,-1)$ we get $\mathbb{A}^{\mathbf{s}} \oplus \mathbb{A}^{-\mathbf{t}} = \mathbb{A}^{(-1,-1)} \oplus \mathbb{A}^{(1,1)}$, the fixed point space of ϕ which is an *associative* algebra (case (1), (ii)); similarly for $\mathbf{t} = (1,1)$, whereas for $\mathbf{t} = (1,-1)$ we have $\mathbb{A}^{\mathbf{s}} \oplus \mathbb{A}^{-\mathbf{t}} = \mathbb{A}^{(-1,1)} \oplus \mathbb{A}^{(1,-1)}$, the antifixed space of the associative automorphism ϕ (case (3), (ii)) (which corresponds to the associative triple system $\mathbb{A}^{-\phi}$).

If neither $\mathbf{s} = \mathbf{t}$ nor $\mathbf{s} = -\mathbf{t}$, then $\mathbb{A}^{\mathbf{s}} \oplus \mathbb{A}^{-\mathbf{t}}$ is equal to one of the spaces $\mathbb{A}^{\tau}$, $\mathbb{A}^{-\tau}$, $\mathbb{A}^{\tilde\tau}$ or $\mathbb{A}^{-\tilde\tau}$ with Lie bracket $[X,Y]_A$, leading to the eight remaining cases of the table. ■

The presentation in form of a table reveals a remarkable duality between "space" (lines) and "deformation parameter" (columns), which is not predicted by the general theory to be developed in Part II (it reminds Howe's duality of dual pairs in some respects). Note that the diagonal terms in the table are all of type "(general linear, half-symplectic)" or "(general linear, unitary)", whereas the antidiagonal terms are algebra cases.

Lemma 3.5 *If τ and $\tilde\tau$ are two commuting involutions and $\phi = \tau \circ \tilde\tau$, then the group $\Gamma := (\mathbb{A}^\phi)^\times$ acts on all four joint eigenspaces by $(g,x) \mapsto gx\tau(g)$. If A and A' are conjugate under this action, then the corresponding homotope LTS are isomorphic to each other.*

Proof. Straightforward calculation (note that $\tau(g) = \tilde\tau(g)$ for $g \in \Gamma$). ■

In all of the following examples, the group Γ turns out to be the "natural" group acting on the given data, so that the description of its orbits amounts in all cases to more or less standard results in linear algebra. This will make classification up to isomorphy quite easy (in most cases, A will be conjugate to an idempotent element under this action). However, if we go beyond the standard examples (for instance, looking at infinite dimensional algebras), then such a classification is generally completely out of reach.

Theorem 3.4 contains a great variety of interesting special cases: indeed, the situation of an associative algebra with two commuting involutions is very common. We are going to work out explicitly some of these special cases.

Remark 3.6 (c-duality.) *In all of the following examples, one may write "c-dual tables" in the following way: consider a real involutive algebra $(\mathbb{B}, *)$ and complexify it: $\mathbb{A} = \mathbb{B}_{\mathbb{C}}$, let τ the $\mathbb{C}$-linear extension of $*$ and $\tilde\tau(X) := \overline{\tau(X)}$ its $\mathbb{C}$-antilinear extension. Then the small squares from the preceding table obtained by taking the middle entries, resp. by the "corner entries", will reproduce the tables for $(\mathbb{B}, *)$ from Lemma 3.2, whereas the other eight entries will contain the c-*dual symmetric pairs *of those. Similarly, if we complexify a real algebra with two commuting involutions, we get a complex algebra with three commuting involutions, and the corresponding table of size* 8×8 *will contain together with each LTS also their c-duals. Replacing $\mathbb{R}$ by $\mathbb{K}$ and $\mathbb{C}$ by $\mathbb{K}[X]/(X^2+1)$, this construction also applies over general base rings $\mathbb{K}$. However, for reasons of space we will not write out such big tables.*

3.3 Case of a matrix algebra with an idempotent

Let $\mathbb{A} = M(n,n;\mathbb{K})$, $\tau(X) = X^t$ (transposed matrix) and $I_{p,q} = \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix}$. These data are paradigmatic for the following, slightly more general, situation: assume given an involutive algebra $(\mathbb{A},\tau)$ with an idempotent e such that $\tau(e) = e$; then $c := 1 - 2e$ is an element such that $c^2 = 1$, and $\tilde{\tau}(x) := c\tau(x)c$ is an involution commuting with τ.

Theorem 3.7 (Homotopes of projective and of polarized spaces) *Let $\mathbb{A} = M(n,n;\mathbb{K})$, fix a decomposition $n = p+q$ and the pair of involutions $(\tau,\tilde{\tau})$ with $\tau(X) = X^t$, $\tilde{\tau}(X) = I_{p,q}X^tI_{p,q}$, so that $\phi(X) = I_{p,q}XI_{p,q}$ is conjugation by $I_{p,q}$. Then the eigenspaces are*

$$\mathbb{A}^{(1,1)} = \Big\{ \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \mid B \in \mathrm{Sym}(p,\mathbb{K}), C \in \mathrm{Sym}(q,\mathbb{K}) \Big\} \cong \mathrm{Sym}(p,\mathbb{K}) \oplus \mathrm{Sym}(q,\mathbb{K})$$

$$\mathbb{A}^{(1,-1)} = \Big\{ \begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix} \mid A \in M(p,q;\mathbb{K}) \Big\} \cong M(p,q;\mathbb{K}),$$

$$\mathbb{A}^{(-1,1)} \cong M(q,p;\mathbb{K}), \quad \mathbb{A}^{(-1,-1)} \cong \mathrm{Asym}(n,\mathbb{K}) \oplus \mathrm{Asym}(n,\mathbb{K})$$

and $\mathbb{A}^{\phi} \cong M(p;\mathbb{K}) \oplus M(q;\mathbb{K})$, $\mathbb{A}^{-\phi} = M(p,q;\mathbb{K}) \oplus M(q,p;\mathbb{K})$. The Lie triple systems from Theorem 3.4 are explicitly given by the following table (where, for the purpose of space economy, we write symmetric pairs in the form of a quotient):

	$A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \in \mathbb{A}^{(1,1)}$	$A \in \mathbb{A}^{(-1,1)}$	$A \in \mathbb{A}^{(1,-1)}$	$A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \in \mathbb{A}^{(-1,-1)}$
$\mathbb{A}^{(1,1)}$	$\frac{\mathfrak{gl}_n(B,\mathbb{K}) \times \mathfrak{gl}_n(B,\mathbb{K})}{\mathfrak{o}_n(B,\mathbb{K}) \times \mathfrak{o}_n(C,\mathbb{K})}$	$\frac{\mathfrak{sp}_{\frac{n}{2}}(\begin{pmatrix} 0 & A \\ -A^t & 0 \end{pmatrix},\mathbb{K})}{\mathfrak{gl}_{p,q}(A,\mathbb{K})}$	$\frac{\mathfrak{sp}_{\frac{n}{2}}(\begin{pmatrix} 0 & A \\ -A^t & 0 \end{pmatrix},\mathbb{K})}{\mathfrak{gl}_{p,q}(A,\mathbb{K})}$	$\mathfrak{sp}_{\frac{n}{2}}(B;\mathbb{K}) \times \mathfrak{sp}_{\frac{n}{2}}(C;\mathbb{K})$
$\mathbb{A}^{(-1,1)}$	$\frac{\mathfrak{o}_n(\begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix},\mathbb{K})}{\mathfrak{o}_n(B;\mathbb{K}) \times \mathfrak{o}_n(C;\mathbb{K})}$	$\mathfrak{gl}_{p,q}(A,\mathbb{K})$	$\mathfrak{gl}_{p,q}(A,\mathbb{K})$	$\frac{\mathfrak{sp}_{\frac{n}{2}}(\begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix},\mathbb{K})}{\mathfrak{sp}_{\frac{n}{2}}(B;\mathbb{K}) \times \mathfrak{sp}_{\frac{n}{2}}(C;\mathbb{K})}$
$\mathbb{A}^{(1,-1)}$	$\frac{\mathfrak{o}_n(\begin{pmatrix} B & 0 \\ 0 & -C \end{pmatrix},\mathbb{K})}{\mathfrak{o}_n(B;\mathbb{K}) \times \mathfrak{o}_n(C;\mathbb{K})}$	$\mathfrak{gl}_{p,q}(A,\mathbb{K})$	$\mathfrak{gl}_{p,q}(A,\mathbb{K})$	$\frac{\mathfrak{sp}_{\frac{n}{2}}(\begin{pmatrix} B & 0 \\ 0 & -C \end{pmatrix},\mathbb{K})}{\mathfrak{sp}_{\frac{n}{2}}(B;\mathbb{K}) \times \mathfrak{sp}_{\frac{n}{2}}(C;\mathbb{K})}$
$\mathbb{A}^{(-1,-1)}$	$\mathfrak{o}_p(B,\mathbb{K}) \times \mathfrak{o}_q(C,\mathbb{K})$	$\frac{\mathfrak{o}_n(\begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix},\mathbb{R})}{\mathfrak{gl}_{p,q}(A,\mathbb{K})}$	$\frac{\mathfrak{o}_n(\begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix},\mathbb{R})}{\mathfrak{gl}_{p,q}(A,\mathbb{K})}$	$\frac{\mathfrak{gl}_n(B,\mathbb{K}) \times \mathfrak{gl}_n(B,\mathbb{K})}{\mathfrak{sp}_{\frac{p}{2}}(B,\mathbb{K}) \times \mathfrak{sp}_{\frac{q}{2}}(C,\mathbb{K})}$

Proof. The determination of the eigenspaces is given by straightforward calculations. Most descriptions of the Lie algebras appearing in the table are also fairly straightforward from definitions, except perhaps the special form of the "middle square": here, the special feature is that there are two diagonal terms $\mathfrak{gl}_{p,q}(A,\mathbb{K})$ of algebra type. This is due to the fact that the Lie algebra $\mathfrak{g}_{\mathbb{A}^{-\phi},A}$ is, in our case, a direct product $\mathfrak{gl}_{p,q}(A,\mathbb{K}) \times \mathfrak{gl}_{p,q}(A,\mathbb{K})$ (indeed, for A as in the table, $\mathbb{A}^{-\phi}$ with $(X,Z) \mapsto XAZ$ is a direct product of associative algebras), and hence we get a LTS of group type. (All this holds, more generally, for associative algebras with idempotent e, as mentioned above, and using the Peirce-decomposition with respect to e.) ∎

Comments. For $\mathbb{K} = \mathbb{R}$, on the level of symmetric spaces, the second and third line of the table describe homotopes of the Grassmannians $\mathrm{Gras}_p(\mathbb{R}^n) = \mathrm{O}(p+q)/\mathrm{O}(p) \times \mathrm{O}(q)$ (which arise for B, C being identity matrices). The first and last line describe homotopes of certain "polarized symmetric spaces" (see Section 4.3 below): notice first that $\mathbb{A}^{(1,1)}$ and $\mathbb{A}^{(-1,-1)}$ have, as vector spaces, a natural direct product structure; the corresponding symmetric spaces inherit this product structure, but whereas for the first and last spaces (in the first and last line) this product structure is *global*, for the middle two spaces it is only *local*: globally, they are not direct products; for $p = q$ they are homotopes of $\mathrm{Sp}(p,\mathbb{R})/\mathrm{Gl}(p,\mathbb{R})$ resp. of $\mathrm{O}(p,p)/\mathrm{Gl}(p,\mathbb{R})$ (which are instances of "Cayley type symmetric spaces"); for $p \neq q$ these families never contain reductive symmetric spaces (and so far seem not to have appeared in the literature).

3.4 Case of the algebra $\mathbb{A} = M(2,2;\mathbb{B})$ for an involutive algebra $\mathbb{B}$

An another important case of application of Theorem 3.4 is the algebra of 2×2-matrices $\mathbb{A} := M(2,2;\mathbb{B})$ with coefficients in $\mathbb{B}$, where $\mathbb{B}$ is an associative algebra with involution $*$. Then $\mathbb{A}$ is again involutive: let us call $\tau_1(X) := (X^*)^t$ ("transposed conjugate matrix") the *standard involution of* $\mathbb{A}$. Besides τ_1, there are at least three other fairly canonical involutions on $\mathbb{A}$. They are defined using the matrices

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad F := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad I := JF = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

For an invertible matrix B let $B_*(X) = BXB^{-1}$ be conjugation by B; if $\tau(B) = \pm B^{-1}$, then $\tau \circ B_* = B_* \circ \tau$ is again an involution. Hence we have the following involutions

$$\tau_J := J_* \circ \tau_1, \quad \tau_F := F_* \circ \tau_1, \quad \tau_I := I_* \circ \tau_1 .$$

These involutions commute among each other, and $\tau_1 \circ \tau_J = J_*$, $\tau_I \circ \tau_F = J_*$, etc. We have the following explicit formulae:

$$\tau_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}, \quad \tau_J \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d^* & -b^* \\ -c^* & a^* \end{pmatrix},$$

$$\tau_F \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d^* & b^* \\ c^* & a^* \end{pmatrix}, \quad \tau_I \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^* & -c^* \\ -b^* & d^* \end{pmatrix}.$$

We call τ_J the *symplectic involution* and τ_F the *artinian involution of* $\mathbb{A}$. In the following, we will mainly be interested in the case $\mathbb{B} = M(n,n;\mathbb{K})$ with $X^* = X^t$ (transposed matrix); then $\mathbb{A} = M(2,2;\mathbb{B}) = M(2n,2n;\mathbb{K})$, and the standard involution τ_1 on $\mathbb{A}$ is precisely the usual transposed of $2n \times 2n$-matrices. Therefore the eigenspaces of τ_1 are $\mathrm{Sym}(2n,\mathbb{K})$ and $\mathrm{Asym}(2n,\mathbb{K})$. For the eigenspaces of τ_F, note that $X^t = X$ is equivalent to $F(FX)^t F = FX$, and hence

$$\mathbb{A}^{\tau_F} = F\mathbb{A}^{\tau_1} = F\mathrm{Sym}(2n,\mathbb{K}), \quad \mathbb{A}^{-\tau_F} = F\mathbb{A}^{-\tau_1} = F\mathrm{Asym}(2n,\mathbb{K}) \tag{4}$$

Similarly, $\mathbb{A}^{\tau_I} = I\mathrm{Sym}(2n,\mathbb{K})$ and $\mathbb{A}^{-\tau_I} = I\mathrm{Asym}(2n,\mathbb{K})$, but

$$\mathbb{A}^{\tau_J} = J\mathbb{A}^{-\tau_1} = J\mathrm{Asym}(2n,\mathbb{K}), \quad \mathbb{A}^{-\tau_J} = J\mathbb{A}^{\tau_1} = J\mathrm{Sym}(2n,\mathbb{K}) \tag{5}$$

In the following, we assume that $\mathbb{K} = \mathbb{R}$. Then $\tau_I \circ \tau_F = J_*$ is conjugation by the standard complex structure on $\mathbb{R}^{2n}$, and hence its fixed point algebra is $M(n,n;\mathbb{C})$.

Theorem 3.8 (Homotopes of the Siegel half plane) *With notation as above, let $\mathbb{B} = M(n,n;\mathbb{R})$, whence $\mathbb{A} = M(2n,2n;\mathbb{R})$, and fix the pair of involutions $(\tau,\tilde{\tau}) := (\tau_I,\tau_F)$, so that $\phi = J_*$ is conjugation by J. Then the eigenspaces are*

$$\mathbb{A}^{(1,1)} = \mathrm{Sym}(n,\mathbb{C}), \quad \mathbb{A}^{(-1,1)} = I\mathrm{Herm}(n,\mathbb{C}), \quad \mathbb{A}^{(1,-1)} = F\mathrm{Herm}(n,\mathbb{C}), \quad \mathbb{A}^{(-1,-1)} = \mathrm{Asym}(n,\mathbb{C})$$

and $\mathbb{A}^\phi = M(n,n;\mathbb{C})$, and the Lie triple systems from Theorem 3.4 are explicitly given by the following table:

	$A \in \mathbb{A}^{(1,1)}$	$A \in \mathbb{A}^{(-1,1)}$	$A \in \mathbb{A}^{(1,-1)}$	$A \in \mathbb{A}^{(-1,-1)}$
$\mathbb{A}^{(1,1)}$	$(\mathfrak{gl}_n(A,\mathbb{C}), \mathfrak{o}_n(A,\mathbb{C}))$	$(\mathfrak{sp}_n(IA,\mathbb{R}), \mathfrak{u}_n(IA,\mathbb{C}))$	$(\mathfrak{sp}_n(FA,\mathbb{R}), \mathfrak{u}_n(FA,\mathbb{C}))$	$\mathfrak{sp}_{\frac{n}{2}}(A,\mathbb{C})$
$\mathbb{A}^{(-1,1)}$	$(\mathfrak{o}_{2n}(IA,\mathbb{R}), \mathfrak{o}_n(A,\mathbb{C})$	$(\mathfrak{gl}_n(IA,\mathbb{C}), \mathfrak{u}_n(IA,\mathbb{C}))$	$\mathfrak{u}_n(FA,\mathbb{C})$	$(\mathfrak{sp}_n(FA,\mathbb{R}), \mathfrak{sp}_{\frac{n}{2}}(A,\mathbb{C}))$
$\mathbb{A}^{(1,-1)}$	$(\mathfrak{o}_{2n}(FA,\mathbb{R}), \mathfrak{o}_n(A,\mathbb{C})$	$\mathfrak{u}_n(IA,\mathbb{C})$	$(\mathfrak{gl}_n(FA,\mathbb{C}), \mathfrak{u}_n(FA,\mathbb{C}))$	$(\mathfrak{sp}_n(IA,\mathbb{R}), \mathfrak{sp}_{\frac{n}{2}}(A,\mathbb{C}))$
$\mathbb{A}^{(-1,-1)}$	$\mathfrak{o}_n(A,\mathbb{C})$	$(\mathfrak{o}_{2n}(FA,\mathbb{R}), \mathfrak{u}_n(IA,\mathbb{C}))$	$(\mathfrak{o}_{2n}(IA,\mathbb{R}), \mathfrak{u}_n(FA,\mathbb{C}))$	$(\mathfrak{gl}_n(A,\mathbb{C}), \mathfrak{sp}_{\frac{n}{2}}(A,\mathbb{C}))$

Proof. First of all, we describe the eigenspaces: as noticed above, $\mathbb{A}^\phi = M(n,n;\mathbb{C})$, that is, a complex matrix $a + ib$ will be identified with $X = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathbb{A}$. Similarly,

$$\mathbb{A}^{-\phi} = \left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \mid a,b \in M(n,n;\mathbb{R}) \right\} = I\,M(n,n;\mathbb{C}) \tag{6}$$

is the space of $\mathbb{C}$-antilinear operators (whose base point I is complex conjugation). As an associative triple system, it is isomorphic to $M(n,n;\mathbb{C})$. Now note that the complex matrix $a+ib$ is symmetric iff a and b are real symmetric matrices, which means that X is fixed under the involution τ_I (and not under the standard involution!). Summing up,

$$\mathrm{Sym}(n,\mathbb{C}) = \mathbb{A}^{J_*} \cap \mathbb{A}^{\tau_I} = \mathbb{A}^{\tau_F} \cap \mathbb{A}^{\tau_I} = \mathbb{A}^{(1,1)} . \tag{7}$$

Similarly

$$\mathbb{A}^{(-1,-1)} = \mathbb{A}^{J_*} \cap \mathbb{A}^{-\tau_I} = \Big\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a^t = -a, b^t = -b \Big\} = \mathrm{Asym}(n,\mathbb{C}).$$

Next, $a+ib$ is Hermitian iff $a = a^t$ and $b = -b^t$, whence $\tau_1(A) = A$ and thus

$$\begin{aligned} \mathbb{A}^{\tau_J} \cap \mathbb{A}^{\tau_1} &= \mathbb{A}^{J_*} \cap \mathbb{A}^{\tau_1} = M(n,n;\mathbb{C}) \cap \mathrm{Sym}(2n;\mathbb{R}) = \mathrm{Herm}(n,\mathbb{C}) \\ \mathbb{A}^{(\tau,-\tilde\tau)} &= \mathbb{A}^{-J_*} \cap \mathbb{A}^{\tau_I} = \Big\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \mid a^t = -a, b^t = b \Big\} = I\,\mathrm{Herm}(n,\mathbb{C}) \\ \mathbb{A}^{(-\tau,\tilde\tau)} &= \mathbb{A}^{-J_*} \cap \mathbb{A}^{-\tau_I} = \Big\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \mid a^t = a, b^t = -b \Big\} = I\,\mathrm{Aherm}(n,\mathbb{C}) = F\,\mathrm{Herm}(n,\mathbb{C}) . \end{aligned} \tag{8}$$

Now, for A belonging to a joint eigenspace, let us determine the Lie triple systems. The Lie triple structure for the four "corners" of the table is simply the one of Lemma 3.2, applied to the algebra $\mathbb{A}^\phi = M(n,n;\mathbb{C})$ with involution being usual transposed. Similarly, the four "inner entries" of the table correspond to the associative pair $\mathbb{A}^{-\phi} = IM(n,n;\mathbb{C})$ with involution corresponding to $X \mapsto \overline{X}^t$ on $M(n,n;\mathbb{C})$. For the remaining eight Lie triple systems $(\mathfrak{g},\mathfrak{h})$, the stabilizer algebra $\mathfrak{h}$ is given by the algebra on the intersection of the same column with the diagonal. The Lie algebra $\mathfrak{g}$ is then one of the algebras $\mathbb{A}^\tau = I\mathrm{Sym}(2n,\mathbb{R})$, $\mathbb{A}^{\tilde\tau} = F\mathrm{Sym}(2n,\mathbb{R})$ (symplectic) or $\mathbb{A}^{-\tau} = I\mathrm{Asym}(2n,\mathbb{R})$, $\mathbb{A}^{-\tilde\tau} = F\mathrm{Asym}(2n,\mathbb{R})$ (orthogonal). One just has to pay attention that all objects are well-defined; for instance, if $A \in I\mathrm{Herm}(n,\mathbb{C})$, then $IA \in \mathrm{Herm}(n,\mathbb{C})$ is a real symmetric matrix (hence $\mathfrak{o}_{2n}(IA;\mathbb{R})$ is well-defined), and $JIA \in J\mathrm{Herm}(n,\mathbb{C}) = \mathrm{Aherm}(n,\mathbb{C})$ is skew-Hermitian (hence $\mathfrak{u}_n(FA;\mathbb{C})$ is well-defined) and also skew-symmetric (hence $\mathfrak{sp}_n(FA,\mathbb{R})$ is well-defined), and so on. ∎

Comments. In the preceding theorem, $\mathbb{A}^{(1,1)}$ and $\mathbb{A}^{(-1,-1)}$ have a natural underlying structure of *complex* vector space, hence on the level of symmetric spaces we obtain homotopes of *(pseudo-) Hermitian symmetric spaces*. In particular, the first line describes homotopes of the Siegel half plane $\mathrm{Sp}_n(\mathbb{R})/\mathrm{U}(n)$. Indeed, the third family of symmetric pairs in this line can be written in the form $(\mathfrak{sp}_n\big(\begin{pmatrix} a & b \\ -b & a \end{pmatrix},\mathbb{R}\big), \mathfrak{u}_n(a+ib,\mathbb{C}))$ with $a \in \mathrm{Asym}(n,\mathbb{R})$ and $b \in \mathrm{Sym}(n,\mathbb{R})$, so that the Siegel half plane corresponds to the choice $a = 0$, $b = 1$.

3.5 Algebras of quaternionic matrices

The next two theorems deal with quaternionic matrices. There are two different viewpoints: we may consider $M(n,n;\mathbb{H})$ as a real form of the complex algebra $\mathbb{A} = M(2n,2n;\mathbb{C})$, thus thinking of quaternionic matrices as a special kind of complex matrices; or we may work intrinsically with matrices having coefficients in $\mathbb{H}$. We start with the latter viewpoint: recall the two involutions of $\mathbb{H}$ from Chapter 2, and let $\mathbb{A} = M(n,n;\mathbb{H})$ and consider the two involutions $\tau(X) := \overline{X}^t$ and $\tilde\tau(X) = \tilde X^t$. They commute, and $\phi = \tau \circ \tilde\tau$ is the automorphism acting on each coefficient by conjugation with the quationion j. The field fixed under conjugation by j in $\mathbb{H}$ is $\mathbb{R} \oplus j\mathbb{R} \cong \mathbb{C}$, and hence $\mathbb{A}^\phi \cong M(n,n;\mathbb{C})$. Note that $\widetilde{(ja_{ij})} = -\widetilde{a}_{ij} j = -j\overline{a}_{ij}$, which implies

$$j\mathrm{Herm}(n,\mathbb{H}) = \mathrm{Aherm}(n,\widetilde{\mathbb{H}}), \quad j\mathrm{Aherm}(n,\mathbb{H}) = \mathrm{Herm}(n,\widetilde{\mathbb{H}}) . \tag{9}$$

Theorem 3.9 (Homotopes of quaternionic type. I) *With notation as above, let $\mathbb{A} = M(n,n;\mathbb{H})$ and fix the pair of involutions $(\tau,\tilde\tau)$. Then the eigenspaces are*

$$\mathbb{A}^{(1,1)} = \mathrm{Herm}(n,\mathbb{C}), \quad \mathbb{A}^{(-1,1)} = i\mathrm{Sym}(n,\mathbb{C}), \quad \mathbb{A}^{(1,-1)} = i\mathrm{Asym}(n,\mathbb{C}), \quad \mathbb{A}^{(-1,-1)} = \mathrm{Aherm}(n,\mathbb{C})$$

and $\mathbb{A}^\phi \cong M(n,n;\mathbb{C})$, and the Lie triple systems from Theorem 3.4 are explicitly given by the following table

	$A \in \mathbb{A}^{(1,1)}$	$A \in \mathbb{A}^{(-1,1)}$	$A \in \mathbb{A}^{(1,-1)}$	$A \in \mathbb{A}^{(-1,-1)}$
$\mathbb{A}^{(1,1)}$	$(\mathfrak{gl}_n(A,\mathbb{C}),\mathfrak{u}_n(A,\mathbb{C}))$	$(\mathfrak{u}_n(A,\widetilde{\mathbb{H}}),\mathfrak{sp}_{\frac{n}{2}}(A,\mathbb{C}))$	$(\mathfrak{u}_n(A,\mathbb{H}),\mathfrak{o}_n(A,\mathbb{C}))$	$\mathfrak{u}_n(A,\mathbb{C})$
$\mathbb{A}^{(-1,1)}$	$(\mathfrak{u}_n(A,\widetilde{\mathbb{H}}),\mathfrak{u}_n(A,\mathbb{C}))$	$(\mathfrak{gl}_n(A,\mathbb{C}),\mathfrak{sp}_{\frac{n}{2}}(A,\mathbb{C}))$	$\mathfrak{o}_n(A,\mathbb{C})$	$(\mathfrak{u}_n(A,\widetilde{\mathbb{H}}),\mathfrak{u}_n(A,\mathbb{C}))$
$\mathbb{A}^{(1,-1)}$	$(\mathfrak{u}_n(A,\mathbb{H}),\mathfrak{u}_n(A,\mathbb{C}))$	$\mathfrak{sp}_{\frac{n}{2}}(A,\mathbb{C})$	$(\mathfrak{gl}_n(A,\mathbb{C}),\mathfrak{sp}_{\frac{n}{2}}(A,\mathbb{C}))$	$(\mathfrak{u}_n(A,\mathbb{H}),\mathfrak{u}_n(A,\mathbb{C}))$
$\mathbb{A}^{(-1,-1)}$	$\mathfrak{u}_n(A,\mathbb{C})$	$(\mathfrak{u}_n(A,\widetilde{\mathbb{H}}),\mathfrak{sp}_{\frac{n}{2}}(A,\mathbb{C}))$	$(\mathfrak{u}_n(A,\mathbb{H}),\mathfrak{o}_n(A,\mathbb{C}))$	$(\mathfrak{gl}_n(A,\mathbb{C}),\mathfrak{u}_n(A,\mathbb{C}))$

Proof. For the eigenspaces, let $A = (a_{ij}) \in M(n,n;\mathbb{H})$. Then $A \in \mathbb{A}^{(1,1)}$ iff $A \in \mathbb{A}^{\phi}$ and $\overline{A}^t = A$, iff $a_{ij} \in \mathbb{R} \oplus j\mathbb{R} = \mathbb{C}$ and $a_{ij} = \overline{a}_{ji}$ in $\mathbb{C}$, that is, iff $A \in \mathrm{Herm}(n,\mathbb{C})$. Similarly, $A \in \mathbb{A}^{(1,-1)}$ iff $a_{ij} \in i\mathbb{R} \oplus k\mathbb{R}$ and $a_{ij} = -a_{ji}$, that is, iff iA is a complex symmetric matrix. The remaining computations are similar as above, so we omit details. ■

Comments. Notice that here the second and third line correspond to homotopes of (pseudo-) Hermitian symmetric spaces. Indeed, by direct inspection we see that the diagram contains precisely the c-dual symmetric pairs from those given in Theorem 3.8. In particular, the third line contains homotopes of the Siegel half plane (or, equivalently, of its compact dual $\mathrm{Sp}(n)/\mathrm{U}(n)$).

Theorem 3.10 (Homotopes of quaternionic type. II) *Let $\mathbb{A} = M(2n,2n;\mathbb{C})$ and fix the pair of involutions $(\tau,\tilde{\tau})$ with $\tau(X) = IX^tI$ and $\tilde{\tau}(X) = F\overline{X}^tF$. Then the eigenspaces are*

$$\mathbb{A}^{(1,1)} \cong \mathrm{Aherm}(n,\mathbb{H}), \quad \mathbb{A}^{(-1,1)} \cong \mathrm{Herm}(n,\mathbb{H}), \quad \mathbb{A}^{(1,-1)} \cong \mathrm{Aherm}(n,\mathbb{H}), \quad \mathbb{A}^{(-1,-1)} = \mathrm{Herm}(n,\mathbb{H})$$

and $\mathbb{A}^{\phi} \cong M(n,n;\mathbb{H})$, and the Lie triple systems from Theorem 3.4 are explicitly given by the following table

	$A \in \mathbb{A}^{(1,1)}$	$A \in \mathbb{A}^{(-1,1)}$	$A \in \mathbb{A}^{(1,-1)}$	$A \in \mathbb{A}^{(-1,-1)}$
$\mathbb{A}^{(1,1)}$	$(\mathfrak{gl}_n(A,\mathbb{H}),\mathfrak{u}_n(A,\widetilde{\mathbb{H}}))$	$(\mathfrak{u}_{2n}(A,\mathbb{C}),\mathfrak{u}_n(A,\mathbb{H}))$	$(\mathfrak{sp}_n(A,\mathbb{C}),\mathfrak{u}_n(A,\widetilde{\mathbb{H}}))$	$\mathfrak{u}_n(A,\mathbb{H})$
$\mathbb{A}^{(-1,1)}$	$(\mathfrak{u}_{2n}(A,\mathbb{C}),\mathfrak{u}_n(A,\widetilde{\mathbb{H}}))$	$(\mathfrak{gl}_n(A,\mathbb{H}),\mathfrak{u}_n(A,\mathbb{H}))$	$\mathfrak{u}_n(A,\widetilde{\mathbb{H}})$	$(\mathfrak{o}_{2n}(A,\mathbb{C}),\mathfrak{u}_n(A,\mathbb{H}))$
$\mathbb{A}^{(1,-1)}$	$(\mathfrak{o}_{2n}(A,\mathbb{C}),\mathfrak{u}_n(A,\widetilde{\mathbb{H}}))$	$\mathfrak{u}_n(A,\mathbb{H})$	$(\mathfrak{gl}_n(A,\mathbb{H}),\mathfrak{u}_n(A,\widetilde{\mathbb{H}}))$	$(\mathfrak{u}_{2n}(A,\mathbb{C}),\mathfrak{u}_n(A,\mathbb{H}))$
$\mathbb{A}^{(-1,-1)}$	$\mathfrak{u}_n(A,\widetilde{\mathbb{H}})$	$(\mathfrak{sp}_n(A,\mathbb{C}),\mathfrak{u}_n(A,\mathbb{H}))$	$(\mathfrak{u}_{2n}(A,\mathbb{C}),\mathfrak{u}_n(A,\widetilde{\mathbb{H}}))$	$(\mathfrak{gl}_n(A,\mathbb{H}),\mathfrak{u}_n(A,\mathbb{H}))$

Proof. Since $\phi(X) = IF\overline{X}FI = J\overline{X}J^{-1}$, the algebra $\mathbb{A}^{\phi}$ is the real form $M(n,n;\mathbb{H})$ of $M(2n,2n;\mathbb{C})$, and $\mathbb{A}^{-\phi} = iM(n,n;\mathbb{H})$ is, as associative triple system, again isomorphic to $M(n,n;\mathbb{H})$. The restriction of τ to $\mathbb{A}^{\phi} = M(n,n;\mathbb{H})$ has the same effect as the involution $X \mapsto \tilde{X}^t$ there, thus $\mathbb{A}^{(1,1)} = \mathrm{Herm}(n,\widetilde{\mathbb{H}}) \cong \mathrm{Aherm}(n,\mathbb{H})$ (mind the isomorphism (9)), and similarly for the other joint eigenspaces. This gives the inner 2×2-square and the square formed by the corner entries. For the remaining eight spaces, observe that $\mathbb{A}^{-\tau} \cong \mathfrak{o}_{2n}(A,\mathbb{C})$ and $\mathbb{A}^{-\tilde{\tau}} \cong \mathfrak{u}_{2n}(A,\mathbb{C})$; combining with the known diagonal entries, this permets to complete the table. ■

4 Homotopes of classical real symmetric spaces

4.1 Classical Groups

The following lemma leads to a definition of Lie groups and algebraic groups corresponding to the algebras defined in Section 2:

Lemma 4.1 *Let $\mathbb{A}$ be an associative algebra with involution $*$ and fix $A \in \mathbb{A}$.*

(1) *The product $X \cdot_A Y = X + Y - XAY$ defines a group structure on the set*

$$G_A := \{X \in \mathbb{A} \mid 1 - XA \in \mathbb{A}^{\times}\}.$$

The neutral element is 0, and the inverse is $j_A(X) = -(1-XA)^{-1}X$. More generally, for any associative pair $(\mathbb{A}^+,\mathbb{A}^-)$ the same formulae define a group structure on $\mathbb{A}^+$, for each $A \in \mathbb{A}^-$.

(2) *If $A^* = A$, then the following is a subgroup of G_A:*

$$U_A := \{X \in G_A \mid j_A(X) = X^*\} = \{X \in G_A \mid X^* + X = X^*AX\}$$

(3) *If $A^* = -A$, then the following is a subgroup of G_A:*

$$S_A := \{X \in G_A \mid j_A(X) = -X^*\} = \{X \in G_A \mid \; X^* - X = X^*AX\}$$

(4) *If $\mathbb{A}$ is finite dimensional over $\mathbb{K} = \mathbb{R}$, then these groups are Lie groups having as Lie algebra the corresponding algebra from Lemma 2.1.*

Proof. (1) Associativity is checked by direct computation; for inversion check first that $X \mapsto 1 - AX$ is a homomorphism from $\cdot_A$ to the usual product. For the case of an associative pair, see [BeKi09].

(2) If $A^* = A$, then $*$ is an antiautomorphism of G_A. Note that the condition $j_A(X) = X^*$ is equivalent to $-X = (1 - XA)X^*$, hence to $X^* + X = XAX^*$. This proves (2), and (3) is shown similarly.

(4) This follows easily by differentiating (see [BeKi09] for a more algebraic argument). ■

If $\mathbb{F}$ is a base ring with involution δ and $\mathbb{A} = M(n, n; \mathbb{F})$ we write also

label	underlying set	parameter space	product
$\mathrm{U}_n(A; \mathbb{F}, \delta)$	$:= \{X \in \mathrm{Gl}_n(A; \mathbb{F}) \vert X + \delta(X)^t = \delta(X)^t AX\}$	$A \in \mathrm{Herm}(n; \mathbb{F}, \delta)$	$X \cdot_A Y$
$\mathrm{Sp}_n(A; \mathbb{F}, \delta)$	$:= \{X \in \mathrm{Gl}_n(A; \mathbb{F}) \vert X - \delta(X)^t = \delta(X)^t AX\}$	$A \in \mathrm{Aherm}(n; \mathbb{F}, \delta)$	$X \cdot_A Y$

and using the standard involutions of $\mathbb{R}, \mathbb{C}, \mathbb{H}$ leads to the following table of classical groups

label	underlying set	parameter space	product
$\mathrm{Gl}_n(A; \mathbb{F})$	$:= \{X \in M(n, n; \mathbb{F}) \vert 1 - AX \text{ invertible}\}$	$A \in M(n, n; \mathbb{F})$	$X \cdot_A Y$
$\mathrm{Gl}_{p,q}(A; \mathbb{F})$	$:= \{X \in M(n, n; \mathbb{F}) \vert 1 - AX \text{ invertible}\}$	$A \in M(q, p; \mathbb{F})$	$X \cdot_A Y$
$\mathrm{O}_n(A; \mathbb{K})$	$:= \{X \in \mathrm{Gl}_n(A, \mathbb{K}) \vert X + X^t = X^t AX\}$	$A \in \mathrm{Sym}(n; \mathbb{K})$	$X \cdot_A Y$
$\mathrm{Sp}_{n/2}(A; \mathbb{K})$	$:= \{X \in \mathrm{Gl}_n(A, \mathbb{K}) \vert X - X^t = X^t AX\}$	$A \in \mathrm{Asym}(n; \mathbb{K})$	$X \cdot_A Y$
$\mathrm{U}_n(A; \mathbb{C})$	$:= \{X \in \mathrm{Gl}_n(A, \mathbb{K}) \vert X + \overline{X}^t = \overline{X}^t AX\}$	$A \in \mathrm{Herm}(n; \mathbb{C})$	$X \cdot_A Y$
$\mathrm{U}_n(A; \mathbb{H})$	$:= \{X \in \mathrm{Gl}_n(A, \mathbb{H}) \vert X + \overline{X}^t = \overline{X}^t AX\}$	$A \in \mathrm{Herm}(n; \mathbb{H})$	$X \cdot_A Y$
$\mathrm{U}_n(A; \widetilde{\mathbb{H}})$	$:= \{X \in \mathrm{Gl}_n(A, \mathbb{H}) \vert X + \widetilde{X}^t = \widetilde{X}^t AX\}$	$A \in \mathrm{Herm}(n; \widetilde{\mathbb{H}})$	$X \cdot_A Y$

Concerning classification up to isomorphy, the same remarks as in Section 2 hold: for all $g, h \in \mathrm{Gl}(n, \mathbb{K})$, the groups $\mathrm{Gl}_n(gAh; \mathbb{K})$ and $\mathrm{Gl}_n(A; \mathbb{K})$ are isomorphic, via the map $X \mapsto gXh$. In particular, $\mathrm{Gl}_n(A; \mathbb{K})$ and $\mathrm{Gl}_n(-A; \mathbb{K})$ are isomorphic. Similarly, unitary or (half-)symplectic groups labelled with A and $A' = gA\tau(g)$ for $g \in \mathrm{Gl}(n, \mathbb{F})$ are isomorphic.

4.2 Classical Symmetric Spaces

Now we are ready to give a list of classical symmetric spaces G/H and their homotopes. The labelling given below corresponds to the classification of Jordan triple systems (see Part II; at this point the labelling may look rather inconsequent – in fact, it follows the one from [Be00], Chapter XII; in particular, 1.1, 1.2, 1.3 are real forms of the complex type 1, and so on). The term “classical symmetric space” is used here for symmetric spaces corresponding to the matrix families of Jordan triple systems. Exceptional spaces and the “semi-exceptional” family of spin factors are not considered here.

Theorem 4.2 *The following tables contain symmetric spaces $M_\alpha = G_\alpha/H_\alpha$ that are homotopes of each other in the following sense: fix an underlying real vector space V^+; in case V^+ is a space $M(p, q; \mathbb{F})$ of rectangular matrices, we let $V^- := M(q, p; \mathbb{F})$, and in all other cases (spaces of symmetric, Hermitian or skew-Hermitian matrices) we let $V^- := V^+$. For each such pair (V^+, V^-) of vector spaces, define families of linear maps $\alpha : V^+ \to V^-$ as in the tables; then V^+ with triple bracket*

$$[X, Y, Z]_\alpha := T(X, \alpha Y, Z) - T(Y, \alpha X, Z),$$

where $T(X, Y, Z) = XYZ + ZYX$ (with XYZ and ZYX being usual matrix products), is a Lie triple system, and it is the LTS belonging to the symmetric space G_α/H_α in the corresponding line of the following tables:

Spaces of rectangular matrices

1. $V^+ = M(p, q; \mathbb{K})$, $V^- = M(q, p; \mathbb{K})$, $\mathbb{K} = \mathbb{R}, \mathbb{C}$:

label	symmetric space G_α/H_α	$\alpha : V^+ \to V^-$	parameter set
1.a	group case $\mathrm{Gl}_{p,q}(A,\mathbb{K})$	$\alpha(X) = AXA$	$A \in M(q,p;\mathbb{K})$
1.a'	$\mathrm{Gl}_{p,q}(A,\mathbb{K}[i])/\mathrm{Gl}_{p,q}(A,\mathbb{K})$	$\alpha(X) = -AXA$	$A \in M(q,p;\mathbb{K})$
1.b	$\mathrm{O}_{p+q}(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix};\mathbb{K})/\mathrm{O}_p(A;\mathbb{K}) \times \mathrm{O}_q(B;\mathbb{K})$	$\alpha(X) = AX^tB$	$A \in \mathrm{Sym}(p,\mathbb{K}), B \in \mathrm{Sym}(q,\mathbb{K})$
1.c	$\mathrm{Sp}_{\frac{p+q}{2}}(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix};\mathbb{K})/\mathrm{Sp}_{\frac{p}{2}}(A;\mathbb{K}) \times \mathrm{Sp}_{\frac{q}{2}}(B;\mathbb{K})$	$\alpha(X) = AX^tB$	$A \in \mathrm{Asym}(p,\mathbb{K}), B \in \mathrm{Asym}(q,\mathbb{K})$

1. cases of $\mathbb{C}$-antilinear α: $V^+ = M(p,q;\mathbb{C})$, $V^- = M(q,p;\mathbb{C})$:

label	symmetric space G_α/H_α	$\alpha : V^+ \to V^-$	parameter set
1.A	$\mathrm{Gl}_{p,q}(A;M(2,2;\mathbb{R}))/\mathrm{Gl}_{p,q}(A;\mathbb{C})$	$\alpha(X) = A\overline{XA}$	$A \in M(q,p;\mathbb{C})$
1.A'	$\mathrm{Gl}_{p,q}(A;\mathbb{H})/\mathrm{Gl}_{p,q}(A;\mathbb{C})$	$\alpha(X) = -A\overline{XA}$	$A \in M(q,p;\mathbb{C})$
1.B	$\mathrm{U}_{p+q}(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix};\mathbb{C})/\mathrm{U}_p(A;\mathbb{C}) \times \mathrm{U}_q(B;\mathbb{C})$	$\alpha(X) = A\overline{X}^tB$	$A \in \mathrm{Herm}(p,\mathbb{C}), B \in \mathrm{Herm}(q,\mathbb{C})$

1.3 $V^+ = M(p,q;\mathbb{H})$, $V^- = M(q,p;\mathbb{H})$

label	symmetric space G_α/H_α	$\alpha : V^+ \to V^-$	parameter set
1.3.a	group case $\mathrm{Gl}_{p,q}(A,\mathbb{H})$	$\alpha(X) = AXA$	$A \in M(q,p;\mathbb{H})$
1.3.a'	$\mathrm{Gl}_{p,q}(A,M(2,2;\mathbb{C}))/\mathrm{Gl}_{p,q}(A,\mathbb{H})$	$\alpha(X) = -AXA$	$A \in M(q,p;\mathbb{H})$
1.3.b	$\mathrm{U}_{p+q}(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix};\mathbb{H})/\mathrm{U}_p(A;\mathbb{H}) \times \mathrm{U}_q(B;\mathbb{H})$	$\alpha(X) = A\overline{X}^tB$	$A \in \mathrm{Herm}(p,\mathbb{H}), B \in \mathrm{Herm}(q,\mathbb{H})$
1.3.c	$\mathrm{U}_{p+q}(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix};\widetilde{\mathbb{H}})/\mathrm{U}_p(A;\widetilde{\mathbb{H}}) \times \mathrm{U}_q(B;\widetilde{\mathbb{H}})$	$\alpha(X) = A\widetilde{X}^tB$	$A \in \mathrm{Herm}(p,\widetilde{\mathbb{H}}), B \in \mathrm{Herm}(q,\widetilde{\mathbb{H}})$

Spaces of symmetric matrices

2. $V := V^+ = V^- = \mathrm{Sym}(n,\mathbb{K})$, $\mathbb{K} = \mathbb{R},\mathbb{C}$:

label	symmetric space G_α/H_α	$\alpha : V \to V$	parameter set
2.a	$\mathrm{Gl}_n(A;\mathbb{K})/\mathrm{O}_n(A;\mathbb{K})$	$\alpha(X) = AXA$	$A \in \mathrm{Sym}(n,\mathbb{K})$
2.a'	$\mathrm{U}_n(A;\mathbb{K}[i])/\mathrm{O}_n(A;\mathbb{K})$	$\alpha(X) = -AXA$	$A \in \mathrm{Sym}(n,\mathbb{K})$
2.b	group space $\mathrm{Sp}_{\frac{n}{2}}(A;\mathbb{K})$	$\alpha(X) = AXA$	$A \in \mathrm{Asym}(n,\mathbb{K})$
2.b'	$\mathrm{Sp}_{\frac{n}{2}}(A;\mathbb{K}[i])/\mathrm{Sp}_{\frac{n}{2}}(A;\mathbb{K})$	$\alpha(X) = -AXA$	$A \in \mathrm{Asym}(n,\mathbb{K})$

2. case of antilinear α: $V = V^+ = V^- = \mathrm{Sym}(n,\mathbb{C})$:

label	symmetric space G_α/H_α	$\alpha : V \to V$	parameter set
2.A	$\mathrm{U}_n(A;\mathbb{H})/\mathrm{U}_n(A;\mathbb{C})$	$\alpha(X) = A\overline{X}A$	$A \in \mathrm{Herm}(n,\mathbb{C})$
2.A'	$\mathrm{Sp}_n(\begin{pmatrix} b & a \\ -a & b \end{pmatrix})/\mathrm{U}_n(b+ia,\mathbb{C})$	$\alpha(X) = -A\overline{X}A$	$A = a+ib \in \mathrm{Herm}(n,\mathbb{C})$

Spaces of skewsymmetric matrices

3. $V = V^+ = V^- = \mathrm{Asym}(n,\mathbb{K})$, $\mathbb{K} = \mathbb{R},\mathbb{C}$:

label	symmetric space G_α/H_α	$\alpha : V \to V$	parameter set
3.a	$\mathrm{Gl}_n(A;\mathbb{K})/\mathrm{Sp}_{n/2}(A;\mathbb{K})$	$\alpha(X) = AXA$	$A \in \mathrm{Asym}(n,\mathbb{K})$
3.a'	$\mathrm{U}_n(A;\mathbb{K}[i])/\mathrm{Sp}_{n/2}(A;\mathbb{K})$	$\alpha(X) = -AXA$	$A \in \mathrm{Asym}(n,\mathbb{K})$
3.b	group case $\mathrm{O}_n(A;\mathbb{K})$	$\alpha(X) = AXA$	$A \in \mathrm{Sym}(n,\mathbb{K})$
3.b'	$\mathrm{O}_n(A;\mathbb{K}[i])/\mathrm{O}_n(A;\mathbb{K})$	$\alpha(X) = -AXA$	$A \in \mathrm{Sym}(n,\mathbb{K})$

3. Case of antilinear α: $V = V^+ = V^- = \mathrm{Asym}(n,\mathbb{C})$

label	symmetric space G_α/H_α	$\alpha : V \to V$	parameter set
3.A	$\mathrm{U}_n(A,\widetilde{\mathbb{H}})/\mathrm{U}_n(A,\mathbb{C})$	$\alpha(X) = A\overline{X}A$	$A \in i\mathrm{Herm}(n,\mathbb{C})$
3.A'	$\mathrm{O}_{2n}(\begin{pmatrix} a & b \\ -b & a \end{pmatrix},\mathbb{R})/\mathrm{U}_n(b+ia,\mathbb{C})$	$\alpha(X) = -A\overline{X}A$	$A = a+ib \in \mathrm{Herm}(n,\mathbb{C})$

Spaces of Hermitian matrices

1.1 $V = V^+ = V^- = \mathrm{Herm}(n,\mathbb{C})$

label	symmetric space G_α/H_α	$\alpha : V \to V$	parameter set
1.1.a	$\mathrm{Gl}_n(A,\mathbb{C})/\mathrm{U}_n(A,\mathbb{C})$	$\alpha(X) = AXA$	$A \in \mathrm{Herm}(n,\mathbb{C})$
1.1.a'	group case $\mathrm{U}_n(A,\mathbb{C})$	$\alpha(X) = -AXA$	$A \in \mathrm{Herm}(n,\mathbb{C})$
1.1.b	$\mathrm{U}_n(A,\widetilde{\mathbb{H}})/\mathrm{O}_n(A,\mathbb{C})$	$\alpha(X) = A\overline{XA}^t$	$A \in \mathrm{Sym}(n,\mathbb{C})$
1.1.b'	$\mathrm{O}_{2n}(\begin{pmatrix} a & b \\ b & -a \end{pmatrix};\mathbb{R})/\mathrm{O}_n(a+ib;\mathbb{C})$	$\alpha(X) = -A\overline{XA}^t$	$A = a+ib \in \mathrm{Sym}(n,\mathbb{C})$
1.1.c	$\mathrm{Sp}_n(\begin{pmatrix} a & b \\ b & -a \end{pmatrix};\mathbb{R})/\mathrm{Sp}_{n/2}(a+ib;\mathbb{C})$	$\alpha(X) = A\overline{XA}^t$	$A = a+ib \in \mathrm{Asym}(n,\mathbb{C})$
1.1.c'	$\mathrm{U}_n(A,\mathbb{H})/\mathrm{Sp}_{n/2}(A,\mathbb{C})$	$\alpha(X) = -A\overline{XA}^t$	$A \in \mathrm{Asym}(n,\mathbb{C})$

3.1 $V^+ = V^- = \mathrm{Herm}(n,\mathbb{H})$:

label	symmetric space G_α/H_α	$\alpha : V \to V$	parameter set
3.1.a	$\mathrm{Gl}_n(A,\mathbb{H})/\mathrm{U}_n(A,\mathbb{H})$	$\alpha(X) = AXA$	$A \in \mathrm{Herm}(n,\mathbb{H})$
3.1.a'	$\mathrm{U}_{2n}(IA,\mathbb{C})/\mathrm{U}_n(A,\mathbb{H})$	$\alpha(X) = -AXA$	$A \in \mathrm{Herm}(n,\mathbb{H})$
3.1.b	group case $\mathrm{U}_n(A,\widetilde{\mathbb{H}})$	$\alpha(X) = A\overline{XA}$	$A \in \mathrm{Herm}(n,\mathbb{H})$
3.1.b'	$\mathrm{O}_{2n}(IA,\mathbb{C})/\mathrm{U}_n(A,\widetilde{\mathbb{H}})$	$\alpha(X) = -A\overline{XA}$	$A \in \mathrm{Herm}(n,\mathbb{H})$

2.2 $V^+ = V^- = \mathrm{Herm}(n,\widetilde{\mathbb{H}})$:

label	symmetric space G_α/H_α	$\alpha : V \to V$	parameter set
2.2.a	$\mathrm{Gl}_n(A,\mathbb{H})/\mathrm{U}_n(A,\widetilde{\mathbb{H}})$	$\alpha(X) = AXA$	$A \in \mathrm{Herm}(n,\widetilde{\mathbb{H}})$
2.2.a'	$\mathrm{U}_{2n}(IA,\mathbb{C})/\mathrm{U}_n(A,\widetilde{\mathbb{H}})$	$\alpha(X) = -AXA$	$A \in \mathrm{Herm}(n,\widetilde{\mathbb{H}})$
2.2.b	group case $\mathrm{U}_n(A,\mathbb{H})$	$\alpha(X) = A\widetilde{\overline{X}}\overline{A}$	$A \in \mathrm{Herm}(n,\widetilde{\mathbb{H}})$
2.2.b'	$\mathrm{Sp}_{2n}(IA,\mathbb{C})/\mathrm{U}_n(A,\mathbb{H})$	$\alpha(X) = -A\widetilde{\overline{X}}\overline{A}$	$A \in \mathrm{Herm}(n,\widetilde{\mathbb{H}})$

Proof of the theorem. All descriptions arise from Theorem 3.4 by identifying a joint eigenspace $\mathbb{A}^{(i,j)}$ with a matrix space V^+. Technical complications arise by the fact that a given matrix space V^+ may be realized in several different ways as a joint eigenspace by using different algebras; this then leads to the various formulas for the endomorphisms α, involving, besides the expression AXA occuring already in Lemma 3.1, transposition and complex or quaternionic conjugation of matrices. When carrying out the computations, one may mind the following general rules:

1. A minus sign always switches from a LTS to its c-dual LTS (indicated by adding a "prime" to the label; if a line contains together with a space its c-dual, then the c-dual line is omitted).

2. For $\mathbb{K} = \mathbb{C}$, a $\mathbb{C}$-linear map α leads to a symmetric space which is defined over $\mathbb{C}$, whereas a $\mathbb{C}$-antilinear map leads to homotopes of (pseudo-)Hermitian symmetric spaces.

3. Every table contains exactly one line which is of "groupe type". For these cases, the LTS can be directly deduced from the definitions (table in Section 2).

4. In all square matrix cases, there is one symmetric space of the form "general linear/unitary (resp. symplectic)", arising from the simpler situation of an algebra with a single involution (Lemma 3.2).

For reasons of place, we will not spell out computations for all cases. Let us just look at some examples: in case 1.b of rectangular matrices we calculate the LTS corresponding to Line 2 of Theorem 3.7: the symmetric space $\mathrm{O}(\mathbb{A},A,\tau)/\mathrm{O}(\mathbb{A},A,\tau) \cap \mathrm{O}(\mathbb{A},A;\tilde{\tau})$ where A is the diagonal matrix with blocks B and C, has LTS $\mathbb{A}^{(-1,1)}$ with triple product $[X,Y,Z]_A$. Now

$$\Big[\begin{pmatrix} 0 & X \\ X^t & 0 \end{pmatrix}, \begin{pmatrix} 0 & Y \\ Y^t & 0 \end{pmatrix}, \begin{pmatrix} 0 & Z \\ Z^t & 0 \end{pmatrix} \Big]_A = \begin{pmatrix} 0 & U \\ U^t & 0 \end{pmatrix},$$

with $U = XBY^tCZ + ZBY^tCX - (YBX^tCZ + ZBX^tCY)$, leading to the claimed formula for the LTS. The computation for the other rectangular case are similar, it suffices to replace τ by the adjoint matrix w.r.t a Hermitian form on the (skew-)fields $\mathbb{C}$ and $\mathbb{H}$.

For the case of the Siegel-space (label 2.A'), the complex conjugation appearing in the formula for α stems from the imbedding $\mathrm{Sym}(n,\mathbb{C}) \subset M(2n,2n;\mathbb{R})$ used in Theorem 3.8; the similar case 3.A' arises from the same theorem. On the other hand, the imbedding $\mathrm{Sym}(n,\mathbb{C}) \subset M(n,n;\mathbb{H})$ from Theorem 3.9 leads to the c-duals of these two cases (labels 2.A and 3.A).

Under the duality of columns and lines mentioned in the preceding chapter, the Siegel-space and its analog correspond to the two families 1.1.b' and 1.1.c having $V = \mathrm{Herm}(n,\mathbb{C})$ as underlying space (Theorem 3.8); similarly for the two families 1.1.b and 1.1.c' using Theorem 3.9.

Finally, tables 3.1 and 2.2 are covered by Theorem 3.10; mind again the isomorphism (9) $\mathrm{Herm}(n,\widetilde{\mathbb{H}}) = j\mathrm{Aherm}(n,\mathbb{H})$ (in the complex case we have, of course, $\mathrm{Aherm}(n,\mathbb{C}) = i\mathrm{Herm}(n,\mathbb{C})$, which explains why table 1.1 corresponding to this case is longer than the others). ■

4.3 Polarized spaces

For the following result, recall that a *twisted polarized symmetric space* is a symmetric space having a local, but not global direct product structure. Formally, such spaces have properties very similar to (pseudo-) Hermitian symmetric spaces, with complex structures ($J^2 = -1$) replaced by polarizations ($J^2 = 1$), see [Be00]. If both eigenspaces of J have equal dimension, one speaks of *para-complex structures* and *para-Hermitian symmetric spaces*. Among the latter are the so-called *Cayley type symmetric spaces* (which correspond precisely to the case when $V^+ = V^- = V$ is a *Euclidean Jordan algebra*). Such spaces always have homotopes that are global direct products; we will not list them here, and focus only on homotopes that are again twisted polarized.

Theorem 4.3 (Homotopes of para-Hermitian symmetric spaces) *The following table contains homotopes of classical para-Hermitian symmetric spaces that are not globally direct products. The underlying space is in all cases $\mathfrak{q} := V^+ \times V^-$, with Lie triple product*

$$[(X,X'),(Y,Y'),(Z,Z')]_\alpha = T((X,X'),\alpha(Y,Y'),(Z,Z')) - T((Y,Y'),\alpha(X,X'),(Z,Z'))$$

with $\alpha : V^+ \times V^- \to V^+ \times V^-$ as below and

$$T((X,X'),(Y,Y'),(Z,Z')) := (XY'Z + ZY'X, X'YZ' + Z'YX').$$

label	symmetric space G_α/H_α	$\alpha : V \to V$	parameter set
1.a	$\mathrm{Gl}_{2p,2q}(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix};\mathbb{K})/\mathrm{Gl}_{p,q}(A;\mathbb{K}) \times \mathrm{Gl}_{p,q}(B;\mathbb{K})$	$\alpha(X,Y) = (AX^tB, A^tY^tB^t)$	$A,B \in M(p,q;\mathbb{K})$
1.b	$\mathrm{Gl}_{p+q}(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix};\mathbb{K})/\mathrm{Gl}_p(A;\mathbb{K}) \times \mathrm{Gl}_q(B;\mathbb{K})$	$\alpha(X,Y) = (AXA, BYB)$	$A \in M(p,p;\mathbb{K})$, $B \in M(q,q;\mathbb{K})$
2.	$\mathrm{Sp}_n(\begin{pmatrix} 0 & A \\ -A^t & 0 \end{pmatrix};\mathbb{K})/\mathrm{Gl}_n(A;\mathbb{K})$	$\alpha(X,Y) = (AXA, AYA)$	$A \in M(n,n;\mathbb{K})$
3.	$\mathrm{O}_{2n}(\begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix};\mathbb{K})/\mathrm{Gl}_n(A;\mathbb{K})$	$\alpha(X,Y) = (AXA, AYA)$	$A \in M(n,n;\mathbb{K})$
1.1.	$\mathrm{U}_{2n}(\begin{pmatrix} 0 & A \\ \overline{A}^t & 0 \end{pmatrix};\mathbb{C})/\mathrm{Gl}_n(A;\mathbb{C})$	$\alpha(X,Y) = (AXA, AYA)$	$A \in M(n,n;\mathbb{C})$
3.1.	$\mathrm{U}_{2n}(\begin{pmatrix} 0 & A \\ \overline{A}^t & 0 \end{pmatrix};\mathbb{H})/\mathrm{Gl}_n(A;\mathbb{H})$	$\alpha(X,Y) = (AXA, AYA)$	$A \in M(n,n;\mathbb{H})$
2.2.	$\mathrm{U}_{2n}(\begin{pmatrix} 0 & A \\ \widetilde{A}^t & 0 \end{pmatrix};\widetilde{\mathbb{H}})/\mathrm{Gl}_n(A;\mathbb{H})$	$\alpha(X,Y) = (AXA, AYA)$	$A \in M(n,n;\mathbb{H})$

Proof. The proof is a special case of the following more general result:

Theorem 4.4 (Homotopes of twisted polarized symmetric spaces) *The following table contains homotopes of classical twisted polarized symmetric spaces that are not globally direct products. Using notation as above,*

label	symmetric space G_α/H_α	(V^+,V^-)	parameter set
1.	$\mathrm{Gl}_{r+s,r'+s'}(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix};\mathbb{K})/\mathrm{Gl}_{r,r'}(A;\mathbb{K})\times \mathrm{Gl}_{s,s'}(B;\mathbb{K})$	$(M(r,s'),M(r',s))$	$A\in M(r',r;\mathbb{K})$, $B\in M(s',s;\mathbb{K})$
2.	$\mathrm{Sp}_n(\begin{pmatrix} 0 & A \\ -A^t & 0 \end{pmatrix};\mathbb{K})/\mathrm{Gl}_{p,q}(A;\mathbb{K})$	$(\mathrm{Sym}(p,\mathbb{K}),\mathrm{Sym}(q,\mathbb{K}))$	$A\in M(q,p);\mathbb{K})$
3.	$\mathrm{O}_{p+q}(\begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix};\mathbb{K})/\mathrm{Gl}_{p,q}(A;\mathbb{K})$	$(\mathrm{Asym}(p;\mathbb{K}),\mathrm{Asym}(q;\mathbb{K}))$	$A\in M(q,p;\mathbb{K})$
1.1.	$\mathrm{U}_{p+q}(\begin{pmatrix} 0 & A \\ \overline{A}^t & 0 \end{pmatrix};\mathbb{C})/\mathrm{Gl}_{p,q}(A;\mathbb{C})$	$(\mathrm{Herm}(p,\mathbb{C}),\mathrm{Herm}(q,\mathbb{C}))$	$A\in M(p,q;\mathbb{C})$
3.1.	$\mathrm{U}_{p+q}(\begin{pmatrix} 0 & A \\ \overline{A}^t & 0 \end{pmatrix};\mathbb{H})/\mathrm{Gl}_{p,q}(A;\mathbb{H})$	$(\mathrm{Herm}(p,\mathbb{H}),\mathrm{Herm}(q,\mathbb{H}))$	$A\in M(q,p;\mathbb{H})$
2.2.	$\mathrm{U}_{p+q}(\begin{pmatrix} 0 & A \\ \widetilde{A}^t & 0 \end{pmatrix};\widetilde{\mathbb{H}})/\mathrm{Gl}_{p,q}(A;\mathbb{H})$	$(\mathrm{Herm}(p,\tilde{\mathbb{H}}),\mathrm{Herm}(q,\tilde{\mathbb{H}}))$	$A\in M(q,p;\mathbb{H})$

Proof. Cases 2 and 3 arise from Theorem 3.7, and 1.1, 3.1 and 2.2 are treated similarly. Proof for case 1: Let $\tau: M(p,q;\mathbb{K})\to M(p,q;\mathbb{K})$, $X\mapsto I_{r,r'}XI_{s,s'}$. Clearly $\tau^2=\mathrm{id}$, and hence, if $\tau(A)=A$ or $\tau(A)=-A$, τ is an automorphism of the Lie triple product $[X,Y,Z]_A$, and hence both eigenspaces are sub-LTS. Now, both eigenspaces are (as vector spaces) direct products of two spaces of rectangular matrices having LTS as given in the claim. ∎

Remark 4.5 *From a Jordan theoretic point of view the latter spaces are all polarized spaces associated to Jordan pairs of the form $(V^+,V^-)=(I,J)$ where (I,J) is a pair of inner ideals in a (simple) Jordan pair (W^+,W^-) (cf. Part II). If I and J are not isomorphic as vector spaces, then the corresponding spaces are polarized but not para-complex, and they are never semi-simple. Nevertheless, they have deformations to semi-simple spaces (namely to direct products of the simple spaces associated to each factor, i.e., to the Jordan pair $(I,I)\times(J,J)$).*

4.4 Isomorphism classes

As pointed out in Lemma 3.5 and the following remark, it is not difficult to deduce from the preceding results a classification up to isomorphism. Generally speaking, isomorphism classes will be parametrized by singling out, for the matrices A,B,C appearing in the parameter spaces, certain matrices in normal form: matrices with coefficients $a_{ii}\in\{0,1\}$, $a_{ij}=0$ else, if A is assumed to be rectangular; diagonal matrices with coefficients $0,1,-1$ if A,B,C are supposed to be symmetric or Hermitian; standard skew matrices if A,B,C are assumed to be skew. Note that the *non-degenerate* choices are exactly those corresponding to the tables from [Be00], Chapter XII; since these lists are already quite long, we will not go into details.

Note also that in *low dimensional cases* several isomorphisms of spaces occur: in Section 3.7 of Part II we give an overview over such isomorphisms.

4.5 Global geometric description: "intrinsic" versus "extrinsic"

The theory explained so far is purely infinitesimal (level of Lie triple systems); the coset spaces G_α/H_α written above are, for the moment, just abstract homogeneous spaces. However, even in the general setting (general base ring $\mathbb{K}$), there are global geometric realizations such that one can "follow the contractions" on the space level. If $\mathbb{K}$ is a topological ring (and in particular for $\mathbb{K}=\mathbb{R}$), the spaces carry manifold structures, and contractions will depend smoothly (in a suitable sense) on the deformation parameter α. We will not go here into details. It is, however, worth mentioning that there are two different kinds of geometric realization, called *intrinsic* and *extrinsic*, respectively (see introduction of [Be00] for an overview). Both of them are well compatible with homotopy.

4.5.1 Intrinsic realization

A simple example is the "general linear group" G_A which, for all A, is realized in Part (1) of Lemma 4.1 as Zariski-dense part of the same underlying space $\mathbb{A}$. In the general case, there is a similar realization, but one has to add “points at infinity” to the underlying affine space of the LTS: for a fixed pair (V^+, V^-), all spaces G_α/H_α can be realized inside a common underlying “generalized projective space” (which can be considered as a “projective completion” of V^+ and is a compact symmetric R-space for the classical real spaces considered here). This can be proved in different ways, where the following strategies 1 and 3 work in the general framework, and strategy 2 works only for $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and suitable topological assumptions (such as finiteness of dimension, or a Banach space context):

1. using Jordan theory, following the general (and abstract) presentation from [Be08];
2. using Lie theory: one notices that $\mathfrak{q}_\alpha := \{v+\alpha v | \, v \in \mathfrak{g}_1\}$ is a “defining plane” in the language of [Ma73], i.e., a sub-LTS of the “conformal” Lie algebra $\mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$ such that the subgroup generated by its exponential image has an open orbit in the projective completion, which then is an open symmetric orbit (this is the original approach by Makarevic, [Ma73], [Ma79], developed further in [Be00]);
3. in the more specific framework of associative algebras with involution from [BeKi09]: as shown there, to an associative algebra corresponds an *associative geometry* $\mathcal{X}$, and involutions τ_i lift to involutions of this geometry. The statements of Lemma 3.1 and Theorem 3.4 then essentially lift to the space level and can thus be used to define symmetric spaces globally.

Conal spaces. For each V^+ there is exactly one family that admits a particularly simple intrinsic description in the underlying vector space V^+ (i.e., like in the example G_A mentioned above, these spaces have “no points at infinity”): these are the spaces of the form G_A/U_A, resp. G_A/S_A associated to an algebra $\mathbb{A}$ with single involution τ. Since the product $X \cdot_A Y = X + Y - XAY$ in $\mathbb{A}$ is associative, the formula

$$G_A \times \mathrm{Herm}(\mathbb{A}) \to \mathrm{Herm}(\mathbb{A}), \quad (g, X) \mapsto g \cdot_A X \cdot_A \tau(g)$$

for $A \in \mathrm{Herm}(\mathbb{A})$ defines an action. The stabilizer of 0 is, by its definition, the τ-unitary group U_A, whence

$$G_A/U_A = \{g \cdot_A g^t \mid g \in G_A\} = \{g + g^t - gAg^t \mid g \in G_A\}$$

which is open in $\mathrm{Herm}(\mathbb{A})$ if $\mathbb{A}$ is real finite-dimensional. As special cases we have global descriptions of the symmetric spaces

$$\mathrm{Gl}_n(A, \mathbb{K})/\mathrm{O}_n(A; \mathbb{K}), \quad \mathrm{Gl}_n(A, \mathbb{F})/\mathrm{U}_n(A; \mathbb{F}), \quad \mathrm{Gl}_n(A, \mathbb{K})/\mathrm{Sp}_{n/2}(A; \mathbb{K})\,,$$

which are contractions of the classical *symmetric cones* (see [FK94]) $M = \mathrm{Gl}(n, \mathbb{R})/\mathrm{O}(n)$, $\mathrm{Gl}(n, \mathbb{C})/\mathrm{U}(n)$, $\mathrm{Gl}(n, \mathbb{H})/\mathrm{Sp}(n)$, and of their non-convex analogs.

4.5.2 Extrinsic realization

The realization of the unitary group U_A from Part (2) of Lemma 4.1 is good example of an *extrinsic symmetric space*: by its definition, inversion j_A in U_A is obtained by restricting the affine map $X \mapsto X^*$ to U_A, and from this it follows easily that the symmetry (inversion) with respect to Y in U_A is induced by the affine map

$$s_Y : \mathbb{A} \to \mathbb{A}, \quad X \mapsto Y \cdot_A X^* \cdot_A Y = 2Y + X^* - YAX^* - YAY - X^*AY + YAX^*AY.$$

An *extrinsic symmetric space* is a submanifold M in an affine space, together with a family $(s_Y)_{Y \in M}$ of affine maps of order 2 and preserving M, called symmetries, such that $s_Y(Y) = Y$ and $s_X s_Y s_X = s_{s_X Y}$, and such that the tangent space $T_Y M$ is exactly the -1-eigenspace of the linear part of s_Y (the $+1$-eigenspace then gives rise to a family of normal spaces; usually, one demands also the existence of a compatible pseudo-Riemannian structure, see, e.g., [Ka08]). These properties are satisfied in the above example of U_A, and similarly for S_A. Note that, as A varies, the submanifold $U_A \subset \mathbb{A}$ also varies, but the family $(U_A)_{A \in \mathbb{A}}$ obviously depends “nicely” on A. In particular, if A tends to zero, this submanifold tends to the affine subspace $\mathrm{Aherm}(\mathbb{A})$ of $\mathbb{A}$. By intersection, all of this generalizes to symmetric spaces defined by two or more involutions of $\mathbb{A}$. The general linear group G_A is not defined by an involution, so it seems not to fit into this picture. But we may realize it in the bigger algebra $\hat{\mathbb{A}} := \mathbb{A} \oplus \mathbb{A}^{op}$ with its exchange involution $\tau((a, b)) = (b, a)$: if $A = (a, a)$, then $U_A(\hat{\mathbb{A}})$ is nothing but G_A, realized as extrinsic symmetric space. All of these facts generalize to the general Jordan theoretic context to be given in Part II: the symmetric spaces G_α/H_α can be realized as families of extrinsic symmetric spaces, such that for $\alpha \to 0$ these submanifolds tend to a flat affine subspace.

4.5.3 Invariant generalized conformal structures

The global intrinsic realization reveals also that all symmetric spaces belonging to the same pair (V^+, V^-) are locally modelled on V^+ and hence have the same underlying invariant "generalized conformal (or projective) structure". This means, for instance:

- if V^+ is a *Euclidean Jordan algebra* ($\mathrm{Sym}(n,\mathbb{R})$, $\mathrm{Herm}(n,\mathbb{C})$, $\mathrm{Herm}(n,\mathbb{H})$), then all homotopes have an invariant *causal structure* (see [HO96]), modelled on the symmetric cone of the Jordan algebra,
- if V^+ is any Jordan algebra (Euclidean or not, such as $M(n,n;\mathbb{F})$ or $\mathrm{Asym}(2n,\mathbb{K})$), then the conal space belonging to $A = 1$ (unit element of the algebra) is invariant under the structure group, and hence gives rise to an invariant conal structure (non-convex in general), which is called a *generalized conformal structure* by Gindikin and Kaneyuki (see [Be00] for more details and references),
- if V^+ is a complex vector space, the homotopes carry an invariant complex structure ("straight" if α is $\mathbb{C}$-linear, and "twisted" in the sense of [Be00] if α is antilinear),
- an invariant polarization if V^+ is as in Section 4.3.

The Jordan theoretic formulation to be given in Part II is well-adapted to describe such structures (and gives, moreover, exceptional examples and the "semi-exceptional" family of projective quadrics).

References

[Berger57] Berger, M, "Les espaces symétriques non-compactes", Ann. Ec. Norm. Sup. **74** (2), p. 85–177

[Be00] Bertram, W., *The geometry of Jordan- and Lie structures*, Springer Lecture Notes in Mathematics **1754**, Berlin 2000

[Be08] Bertram, W., "Homotopes and conformal deformations of symmetric spaces." J. Lie Theory **18** (2008), 301 – 333 (arxiv: math.RA/0606449)

[BeBi] Bertram, W. and Bieliavsky, P., "Homotopes of symmetric spaces. II: Structure variety and classification" (Part II of the present work, arxiv, november 2011)

[BeKi09] BeKi09b Bertram, W. and M. Kinyon, "Associative Geometries. I: Torsors, Linear Relations and Grassmannians. II: Involutions, the Classical Torsors, and their Homotopes" , J. Lie Theory **20** (2) (2010), 215–252 and 253–282 (arXiv : math.RA/0903.5441 and math.RA/0909.4438)

[BDS09] Bieliavsky, P., S. Detournay and Ph. Spindel, "The Deformation Quantization of the Hyperbolic Plane", Com. Math. Phys. **289** (2), (2009), 529 – 559

[FK94] Faraut, J. and A. Koranyi, *Analysis on Symmetric Cones*, Clarendon Press, Oxford 1994

[HO96] Hilgert, J. and G. Olafsson, *Causal Symmetric Spaces - Geometry and Analysis* Birkhäuser, 1996

[Ka08] Kath, I, "Indefinite extrinsic symmetric spaces. I", arxiv: 0809.4713v2 [math.DG]

[Lo69] Loos, O., *Symmetric Spaces I*, Benjamin, New York, 1969

[Lo75] Loos, O., *Jordan Pairs*, Springer LNM, 1975

[Ma73] Makarevic, B.O., Open symmetric orbits of reductive groups in symmetric R-spaces. Math. USSR Sbornik **20** (1973), 406 – 418

[Ma79] Makarevic, B.O., Jordan algebras and orbits in symmetric R-spaces. Trans. Moscow. Math. Soc. **39** (1979), 169 – 192