COORDINATE ALGEBRAS OF EXTENDED AFFINE LIE ALGEBRAS OF TYPE $\mathbf{A}_1$

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Abstract. The cores of extended affine Lie algebras of reduced types were classified except for type A_1. In this paper we determine the coordinate algebra of extended affine Lie algebras of type A_1. It turns out that such an algebra is a unital $\mathbb{Z}^n$-graded Jordan algebra of a certain type, called a *Jordan torus.* We classify Jordan tori and get five types of Jordan tori.

Introduction

Extended affine Lie algebras form a new class of infinite dimensional Lie algebras, which were first introduced by Høegh-Krohn and Torresani in 1990 [7] (under the name of irreducible quasi-simple Lie algebras) as a generalization of the finite dimensional simple Lie algebras and the affine Kac-Moody Lie algebras, and systematically studied in the recent memoir [1].

Roughly speaking, an extended affine Lie algebra, EALA for short, is a complex Lie algebra which has a nondegenerate symmetric invariant form, a self-centralizing finite dimensional ad-diagonalizable abelian subalgebra and a discrete irreducible root system such that elements from non-isotropic root spaces are ad-nilpotent. The *core* of an EALA is defined as the subalgebra generated by the non-isotropic root spaces.

One has a description of an EALA $\mathcal{L}$ of type A_l ($l \geq 2$), D_l and E_l due to Berman, Gao and Krylyuk [5]. Their description is a 2-step process:

A) describe the core $\mathcal{L}_c$, and then,

B) describe how $\mathcal{L}_c$ sits in $\mathcal{L}$.

This program is currently being worked out for the other types of EALA's. In particular, Allison and Gao [2] describe the cores of all non-simply laced reduced types, i.e., B_l, C_l, G_2 and F_4. In this paper we describe the cores of the only missing case in reduced types, namely, EALA's of type A_1.

For motivation, we recall the definition of quantum tori and then describe the core of an EALA of type A_l, $l \geq 2$.

Definition. An $n \times n$ matrix $\boldsymbol{q} = (q_{ij})$ over a field F such that $q_{ii} = 1$ and $q_{ji} = q_{ij}^{-1}$ is called a *quantum matrix.* The *quantum torus* $F_{\boldsymbol{q}} = F_{\boldsymbol{q}}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ *determined by* a quantum matrix $\boldsymbol{q}$ is defined as the associative algebra over F with $2n$ generators $t_1^{\pm 1}, \dots, t_n^{\pm 1}$, and relations $t_i t_i^{-1} = t_i^{-1} t_i = 1$ and $t_j t_i = q_{ij} t_i t_j$ for all $1 \leq i, j \leq n$. Note that $F_{\boldsymbol{q}}$ is commutative if and only if $\boldsymbol{q} = \mathbf{1}$ where $\mathbf{1}$ is the quantum matrix whose entries are all 1. In this case, the quantum torus $F_{\mathbf{1}}$ becomes the algebra of Laurent polynomials $F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ in n variables.

A quantum torus is characterized as a unital Λ-graded associative algebra $A = \oplus_{\boldsymbol{\alpha} \in \Lambda} A_{\boldsymbol{\alpha}}$ over F satisfying

(1) $A_{\boldsymbol{\alpha}} A_{\boldsymbol{\beta}} = A_{\boldsymbol{\alpha}+\boldsymbol{\beta}}$ for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Lambda$, i.e., A *is strongly graded*,

(2) $\dim_F A_{\boldsymbol{\alpha}} = 1$ for all $\boldsymbol{\alpha} \in \Lambda$.

Note that all nonzero homogeneous elements of A are invertible.

Let

$$\mathrm{sl}_{l+1}(F_{\boldsymbol{q}}) := \{X \in M_{l+1}(F_{\boldsymbol{q}}) \mid \mathrm{tr}(X) \in [F_{\boldsymbol{q}}, F_{\boldsymbol{q}}]\}$$

be the subalgebra of the Lie algebra $M_{l+1}(F_{\boldsymbol{q}})$ over F of $(l+1) \times (l+1)$ matrices over $F_{\boldsymbol{q}}$ where $\mathrm{tr}(X)$ is the trace of X and $[F_{\boldsymbol{q}}, F_{\boldsymbol{q}}]$ is the span of all commutators $[a, b] = ab - ba$. It is shown in [5] that the core of any EALA of type A_l, $l \geq 3$, is a central extension of $\mathrm{sl}_{l+1}(\mathbb{C}_{\boldsymbol{q}})$ for $F = \mathbb{C}$, the field of complex numbers.

The central extensions of the Lie algebra $\mathrm{sl}_3(\mathbb{C}_{\boldsymbol{q}})$ are examples for cores of an EALA of type A_2, but do not give all possibilities. Rather, there exists a construction which associates to every alternative algebra A a Lie algebra $\mathrm{psl}_3(A)$, and it is shown in [6] that a core of an EALA of type A_2 is a central extension of $\mathrm{psl}_3(A)$ where A is a unital Λ-graded alternative algebra $A = \oplus_{\boldsymbol{\alpha} \in \Lambda} A_{\boldsymbol{\alpha}}$ over $\mathbb{C}$ satisfying (1) and

(2) above. These alternative algebras have been classified in [6]. Besides $\mathbb{C}_{\boldsymbol{q}}$, there exists up to isomorphisms one more type, the *Cayley torus* $\mathbb{O}_t$. It is defined as $\mathbb{O}_t = (\mathbb{C}[t_1^{\pm1}, \dots, t_n^{\pm1}], t_1, t_2, t_3)$, i.e., the octonion algebra over $\mathbb{C}[t_1^{\pm1}, \dots, t_n^{\pm1}]$ obtained by the Cayley-Dickson process with the structure constants t_1, t_2 and t_3.

Now, for EALA's of type A_1, the Tits-Kantor-Koecher construction which associates to every Jordan algebra J a Lie algebra $\mathrm{TKK}(J)$, called the *TKK algebra of J* (see e.g. [8]), comes into play. We show:

Theorem 1. *The core of any EALA of type A_1 is a central extension of $\mathrm{TKK}(J)$ where J is a unital Λ-graded Jordan algebra $J = \oplus_{\boldsymbol{\alpha}\in\Lambda} J_{\boldsymbol{\alpha}}$ over $\mathbb{C}$ satisfying*

(T1) $\{\boldsymbol{\alpha} \in \Lambda \mid J_{\boldsymbol{\alpha}} \neq (0)\}$ *generates* Λ,
(T2) *all nonzero homogeneous elements are invertible,*
(T3) $\dim_{\mathbb{C}} J_{\boldsymbol{\alpha}} \leq 1$ *for all* $\boldsymbol{\alpha} \in \Lambda$.

Such a graded Jordan algebra over any field F of characteristic $\neq 2$ is called a *Jordan n-torus* or simply a *Jordan torus*. We classify Jordan tori not only over $\mathbb{C}$ but over F. The simplest example of Jordan tori over F is the plus algebra $F_{\boldsymbol{q}}^+$ of a quantum torus $F_{\boldsymbol{q}}$, which is defined on the space $F_{\boldsymbol{q}}$ with the new product $x \cdot y := \frac{1}{2}(xy + yx)$ for $x, y \in F_{\boldsymbol{q}}$. We note that $\mathrm{sl}_2(F_{\boldsymbol{q}}) \cong \mathrm{TKK}(F_{\boldsymbol{q}}^+)$. To state our result, we briefly describe other examples of Jordan n-tori over F.

(a) Let $\boldsymbol{\varepsilon} = (\varepsilon_{ij})$ be a quantum matrix such that $\varepsilon_{ij} = 1$ or -1 for all i, j. Define an involution $*$ on $F_{\boldsymbol{\varepsilon}} = F_{\boldsymbol{\varepsilon}}[t_1^{\pm1}, \dots, t_n^{\pm1}]$ such that $t_i^* = t_i$ for all i. Then the symmetric elements $H(F_{\boldsymbol{\varepsilon}}, *)$ form a Jordan torus. It is a subalgebra of $F_{\boldsymbol{\varepsilon}}^+$.

Also, let E be a quadratic field extension of F with the nontrivial Galois automorphism σ_E. Let $\boldsymbol{\xi} = (\xi_{ij})$ be a quantum matrix such that $\xi_{ij}\sigma_E(\xi_{ij}) = 1$ for all i, j. Define a σ_E-semilinear involution σ on $E_{\boldsymbol{\xi}} = E_{\boldsymbol{\xi}}[t_1^{\pm1}, \dots, t_n^{\pm1}]$ over F such that $\sigma(t_i) = t_i$ for all i. Then the symmetric elements $H(E_{\boldsymbol{\xi}}, \sigma)$ form a Jordan torus over F. It is an F-subalgebra of $E_{\boldsymbol{\xi}}^+$.

(b) Let $2 \leq m \leq n$ and let $S^{(m)}$ be any *semilattice* in $\mathbb{Z}^m$ (see 1.2 for the precise definition). One can construct a Jordan algebra $J_{S^{(m)}}(\{a_{\boldsymbol{\epsilon}}\}_{\boldsymbol{\epsilon}\in I})$ over $F[t_1^{\pm1}, \dots, t_n^{\pm1}]$ of a certain symmetric bilinear form which depends on $S^{(m)}$ and a family of nonzero elements $a_{\boldsymbol{\epsilon}} \in F$ indexed by some set I (the details are in 5.2). It turns out that $J_{S^{(m)}}(\{a_{\boldsymbol{\epsilon}}\}_{\boldsymbol{\epsilon}\in I})$ is a Jordan n-torus called a *Clifford torus*. Clifford tori are a slight generalization of a construction which already appeared in [1].

(c) Suppose that F contains a primitive 3rd root of unity ω. Let $\boldsymbol{\omega} = (\omega_{ij})$ be a quantum matrix such that $\omega_{12} = \omega$, $\omega_{21} = \omega^2$ and $\omega_{ij} = 1$ for the other i, j. Let $\mathbb{A}_t =$

$(F_{\boldsymbol{\omega}}, t_3)$ be the first Tits construction, using the quantum torus $F_{\boldsymbol{\omega}} = F_{\boldsymbol{\omega}}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ and the structure constant t_3 (details are in 6.8). The central closure of $\mathbb{A}_t$ is a 27-dimensional exceptional Jordan division algebra over a rational function field in n variables. We will see that $\mathbb{A}_t$ is a Jordan torus, which is called the *Albert torus*. This torus was independently found in [1] and [20]. It is a coordinate algebra of EALA's of type G_2 (see [1] and [2]).

We can now state our main result:

Theorem 2. *Let J be a Jordan torus over F. Then J is isomorphic to one of the five tori*

$$F_{\boldsymbol{q}}^+, \ H(F_{\boldsymbol{\varepsilon}}, *), \ H(E_{\boldsymbol{\xi}}, \sigma), \ J_{S^{(m)}}(\{a_{\boldsymbol{\epsilon}}\}_{\boldsymbol{\epsilon} \in I}) \quad or \quad \mathbb{A}_t.$$

Since Jordan tori turn out to be strongly prime, we can use Zelmanov's Prime Structure Theorem [14] as the first step of our proof. Thus, a Jordan torus is either of Hermitian, Clifford or Albert type. For each type we then determine the possible Jordan tori.

This paper consists of 7 sections: In §1 we give the definition of an EALA and prove Theorem 1 above. In §2 some basic concepts of Jordan algebras are reviewed. In §3 general properties of Jordan tori are described. In §4 we show that a Hermitian torus is graded isomorphic to $F_{\boldsymbol{q}}^+$, $H(F_{\boldsymbol{\varepsilon}}, *)$ or $H(E_{\boldsymbol{\xi}}, \sigma)$. In §5 we show that any Clifford torus is graded isomorphic to $J_{S^{(m)}}(\{a_{\boldsymbol{\epsilon}}\}_{\boldsymbol{\epsilon} \in I})$. In §6 the special Jordan tori of central degree 3 are first determined. Then we prove that any Jordan torus of Albert type is graded isomorphic to the Albert torus $\mathbb{A}_t$. As a corollary, we obtain the classification of Jordan tori of central degree 3, which is used in the classification of cores of EALA's of type G_2 in [2]. In §7 the classification of Jordan tori are summarized.

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§1 Extended affine Lie algebras of type A_1

We define extended affine Lie algebras [1]. Let $\mathcal{L}$ be a Lie algebra over $\mathbb{C}$ (the field of complex numbers). Assume that

(EA1) $\mathcal{L}$ has a nondegenerate invariant symmetric bilinear form $(\cdot, \cdot)$.
Here 'invariant' means that $(\cdot, \cdot)$ satisfies $([x, y], z) = (x, [y, z])$ for all $x, y, z \in \mathcal{L}$.

(EA2) $\mathcal{L}$ has a nontrivial finite dimensional self-centralizing ad-diagonalizable abelian subalgebra $\mathcal{H}$.

We will be assuming three further axioms about the triple $(\mathcal{L}, (\cdot,\cdot), \mathcal{H})$. To describe them we need some further notation. Because of (EA2), we have

$$\mathcal{L} = \oplus_{\alpha\in\mathcal{H}^*} \mathcal{L}_\alpha \quad \text{and} \quad \mathcal{L}_0 = \mathcal{H}$$

where $\mathcal{H}^*$ is the complex dual space of $\mathcal{H}$ and for any $\alpha \in \mathcal{H}^*$ we define $\mathcal{L}_\alpha = \{x \in \mathcal{L} \mid [h,x] = \alpha(h)x \text{ for all } h \in H\}$. Let $R = \{\alpha \in \mathcal{H}^* \mid \mathcal{L}_\alpha \neq (0)\}$. R is called the *root system* of $\mathcal{L}$. Note that since $\mathcal{H} \neq (0)$, we have $0 \in R$. Also, $\alpha, \beta \in R, \alpha+\beta \neq 0 \Longrightarrow (\mathcal{L}_\alpha, \mathcal{L}_\beta) = \{0\}$. Thus, $-R = R$ by nondegeneracy. Moreover, $(\cdot,\cdot)$ is nondegenerate on $\mathcal{H}$. As in the classical theory of finite-dimensional complex semisimple Lie algebras, we can transfer $(\cdot,\cdot)$ to a form on $\mathcal{H}^*$. Let

$$R^\times = \{\alpha \in R \mid (\alpha,\alpha) \neq 0\} \quad \text{and} \quad R^0 = \{\alpha \in R \mid (\alpha,\alpha) = 0\}.$$

The elements of $R^\times$ (resp. R^0) are called *non-isotropic* (resp. *isotropic*) roots. We have $R = R^\times \cup R^0$. We further require that

(EA3) $\alpha \in R^\times$, $x_\alpha \in \mathcal{L}_\alpha \Longrightarrow \operatorname{ad} x_\alpha$ acts locally nilpotently on $\mathcal{L}$.

(EA4) R is a discrete subset of $\mathcal{H}^*$.

(EA5) R is irreducible. That is,

(a) $R^\times = R_1 \cup R_2$, $(R_1, R_2) = (0) \Longrightarrow R_1 = \emptyset$ or $R_2 = \emptyset$

(b) $\sigma \in R^0 \Longrightarrow$ there exists $\alpha \in R^\times$ such that $\alpha + \sigma \in R^\times$.

If $\mathcal{L}$ satisfies (EA1)-(EA5), the triple $(\mathcal{L}, (\cdot,\cdot), \mathcal{H})$, or simply the algebra $\mathcal{L}$ itself, is called an *extended affine Lie algebra* or EALA for short.

Let t_α be the unique element of $\mathcal{H}$ so that $(t_\alpha, h) = \alpha(h)$ for $h \in \mathcal{H}$, and put $h_\alpha = \frac{2}{(\alpha,\alpha)} t_\alpha$. Then there exist nonzero $e_\alpha \in \mathcal{L}_\alpha$ and $f_\alpha \in \mathcal{L}_{-\alpha}$ so that $[h_\alpha, e_\alpha] = 2e_\alpha$, $[h_\alpha, f_\alpha] = -2f_\alpha$ and $[e_\alpha, f_\alpha] = h_\alpha$. In other words, $\{e_\alpha, h_\alpha, f_\alpha\}$ is an sl_2-triplet. Thus we can use sl_2-theory. Assuming only (EA1), (EA2) and (EA3), one can show that some well-known properties of finite dimensional semisimple Lie algebras over $\mathbb{C}$ are also true for EALA's.

1.1. ([1], Lemma I.1.21 and Theorem I.1.29) *Let $\alpha \in R^\times$, $\beta \in R$ and $r = 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}$. Then:*

(i) $2\frac{(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.

(ii) $\dim_{\mathbb{C}} \mathcal{L}_\alpha = 1$.

(iii) *Assume that* $\operatorname{ad} e_\alpha(e_\beta) = 0$. *Then* $r \geq 0$,

$$(\operatorname{ad} f_\alpha)^i(e_\beta) \neq 0, \quad \textit{for all} \quad i = 0, 1, \ldots, r, \quad \textit{and} \quad (\operatorname{ad} f_\alpha)^{r+1}(e_\beta) = 0. \quad \square$$

In the following, $\mathcal{L}$ is an EALA with root system R. We recall some of the properties of R that we will need. First, note that if a nondegenerate invariant symmetric bilinear form on $\mathcal{L}$ is multiplied by a nonzero complex number, then we still have such a form. Since the axioms are invariant under such a change, we may as well assume that there is some non-isotropic root $\alpha \in R^\times$ with $(\alpha, \alpha) \in \mathbb{R}_{>0}$. Then if $\beta \in R^\times$ we have $2\frac{(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$ so that $(\beta, \alpha) \in \mathbb{R}$, and hence, since $2\frac{(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$ we get $(\beta, \beta) \in \mathbb{R}$ if $(\alpha, \beta) \neq 0$. It now follows, using (EA5)(a), that $(\alpha, \beta) \in \mathbb{R}$ for any $\alpha, \beta \in R$. That is, our form is real valued on the real linear span of the roots. From now on *we assume that our form is scaled so that there is at least one* $\alpha \in R^\times$ *with* $(\alpha, \alpha) > 0$. Let $\mathcal{V}$ be the real span of R in $\mathcal{H}$. Then it was proven in [1] Theorem I.2.14 that the real valued symmetric bilinear form $(\cdot, \cdot)\,|_{\mathcal{V}}$ is positive semidefinite on $\mathcal{V}$. This was a conjecture of Kac. Let

$$\mathcal{V}^0 = \{\alpha \in \mathcal{V} \mid (\alpha, \beta) = 0 \text{ for all } \beta \in \mathcal{V}\}.$$

The *nullity* of R or of $\mathcal{L}$ is defined to be the real dimension n of $\mathcal{V}^0$. Let $\overline{\mathcal{V}} = \mathcal{V}/\mathcal{V}^0$ and let $^- : \mathcal{V} \longrightarrow \overline{\mathcal{V}}$ be the canonical projection. Then $(\cdot, \cdot)$ induces a positive definite symmetric bilinear form on $\overline{\mathcal{V}}$ so that, relative to this form, the image $\overline{R}$ of R in $\overline{\mathcal{V}}$ is a finite irreducible (possibly non-reduced) root system [1] Proposition I.2.19. The *type* of R or of $\mathcal{L}$ is defined to be the type of the finite root system $\overline{R}$.

Our interest is in EALA's of type A_1. Thus $\overline{R} = \{\overline{0}, \pm\overline{\alpha}\}$. We choose a fixed preimage $\dot{\alpha} \in R$ of $\overline{\alpha}$ under $^-$. Let $\dot{\mathcal{V}}$ be the subspace spanned by $\dot{\alpha}$ of $\mathcal{V}$. Then we have $\mathcal{V} = \dot{\mathcal{V}} \oplus \mathcal{V}^0$, and $^-$ restricts to an isometry of $\dot{\mathcal{V}}$ onto $\overline{\mathcal{V}}$. Let $\dot{R}$ be the image of R under the projection of $\mathcal{V}$ onto $\dot{\mathcal{V}}$, and so $\dot{R} = \{0, \pm\dot{\alpha}\}$. We define

$$S = \{\sigma \in \mathcal{V}^0 \mid \dot{\alpha} + \sigma \in R\} \quad \text{and} \quad \Lambda = \text{ the subgroup of } \mathcal{V}^0 \text{ generated by } R^0.$$

Then, by [1] Chapter II, we have $R^0 = S + S$ and $R = (S + S) \cup (\dot{\alpha} + S) \cup (-\dot{\alpha} + S)$. Moreover, S is a discrete spanning set in $\mathcal{V}^0$, Λ is a lattice in $\mathcal{V}^0$ and S is a semilattice in Λ:

Definition 1.2. A subset S of a lattice Λ which has the following three properties,

(i) $0 \in S$,
(ii) $2\sigma - \tau \in S$ for all $\sigma, \tau \in S$,
(iii) S generates Λ,

is called a *semilattice in* Λ.

We put

$$\mathcal{L}^\sigma := \mathcal{L}_{-\dot{\alpha}+\sigma} \oplus \mathcal{L}_\sigma \oplus \mathcal{L}_{\dot{\alpha}+\sigma} \quad \text{for} \quad \sigma \in \Lambda.$$

Then we have $\mathcal{L} = \oplus_{\sigma\in\Lambda}\ \mathcal{L}^\sigma$ and $[\mathcal{L}^\sigma, \mathcal{L}^\tau] \subset \mathcal{L}^{\sigma+\tau}$ for $\sigma, \tau \in \Lambda$. In other words, $\mathcal{L}$ is a Λ-graded Lie algebra.

Definition 1.3. The *core* of $\mathcal{L}$ is defined to be the subalgebra $\mathcal{L}_c$ of $\mathcal{L}$ generated by the spaces $\mathcal{L}_\alpha$, $\alpha \in R^\times$.

Since $\mathcal{L}_c$ is generated by homogeneous elements, $\mathcal{L}_c$ is a Λ-graded subalgebra of $\mathcal{L}$. Thus, $\mathcal{L}_c = \oplus_{\sigma\in\Lambda}\ (\mathcal{L}_c)^\sigma$, where for $\sigma \in \Lambda$, and

$$(\mathcal{L}_c)^\sigma := \mathcal{L}_c \cap \mathcal{L}^\sigma = \mathcal{L}_{-\dot{\alpha}+\sigma} \oplus \sum_{\tau,\nu\in\Lambda,\tau+\nu=\sigma} [\mathcal{L}_{\dot{\alpha}+\tau}, \mathcal{L}_{-\dot{\alpha}+\nu}] \oplus \mathcal{L}_{\dot{\alpha}+\sigma}.$$

Also, let

$$(\mathcal{L}_c)_{-\dot{\alpha}} := \oplus_{\sigma\in\Lambda}\ \mathcal{L}_{-\dot{\alpha}+\sigma}, \quad (\mathcal{L}_c)_{\dot{\alpha}} := \oplus_{\sigma\in\Lambda}\ \mathcal{L}_{\dot{\alpha}+\sigma}, \quad (\mathcal{L}_c)_0 := \sum_{\tau,\nu\in\Lambda} [\mathcal{L}_{\dot{\alpha}+\tau}, \mathcal{L}_{-\dot{\alpha}+\nu}].$$

Then we have $\mathcal{L}_c = (\mathcal{L}_c)_{-\dot{\alpha}} \oplus (\mathcal{L}_c)_0 \oplus (\mathcal{L}_c)_{\dot{\alpha}}$. Let $\dot{\mathcal{G}} := \langle e_{\dot{\alpha}}, h_{\dot{\alpha}}, f_{\dot{\alpha}}\rangle \cong sl_2(\mathbb{C})$ and $\dot{\mathcal{H}} := \mathbb{C}h_{\dot{\alpha}}$, which are both subalgebras of $\mathcal{L}_c$. Obviously, for each $\varepsilon = 0, -1, 1$

$$(\mathcal{L}_c)_{\varepsilon\dot{\alpha}} = \{x \in \mathcal{L}_c \mid [h, x] = \varepsilon\dot{\alpha}(h)x \text{ for all } h \in \dot{\mathcal{H}}\},$$

and $\mathcal{L}_c$ is generated as an algebra by $(\mathcal{L}_c)_{-\dot{\alpha}}$ and $(\mathcal{L}_c)_{\dot{\alpha}}$. Therefore, $\mathcal{L}_c$ is graded by the root system of type A_1 as defined in [4]. By the description of such Lie algebras (see [3] or [15]), $\mathcal{L}_c$ is a central extension of the TKK algebra of a unital Jordan algebra. The Jordan algebra J, called the *coordinate algebra of $\mathcal{L}$ of type A_1*, is defined as follows. Let $J := (\mathcal{L}_c)_{\dot{\alpha}}$ as a $\mathbb{C}$- vector space and define a multiplication on J by

$$xy := \frac{1}{2}\big[[x, f_{\dot{\alpha}}], y\big] \quad \text{for} \quad x, y \in J.$$

One can check that this multiplication is commutative and satisfies the Jordan identity, and so J is a Jordan algebra over $\mathbb{C}$. Note that $e_{\dot{\alpha}}$ is the identity element of J. Our goal is to describe the structure of J. We put $J_\sigma := \mathcal{L}_{\dot{\alpha}+\sigma}$. Then $1 := e_{\dot{\alpha}} \in J_0$, and one can easily see that $J = \oplus_{\sigma\in\Lambda}\ J_\sigma$ is a Λ-graded Jordan algebra over $\mathbb{C}$, i.e., $J_\sigma J_\tau \subset J_{\sigma+\tau}$ for all $\sigma, \tau \in \Lambda$. Also, by 1.1(ii), we have

$$\dim_{\mathbb{C}} J_\sigma = 1 \quad \text{if} \quad \sigma \in S \quad \text{and} \quad J_\sigma = (0) \quad \text{if} \quad \sigma \in \Lambda \setminus S.$$

These conditions are not enough to classify J. To obtain a crucial property of J, 'invertibility of nonzero homogeneous elements', we use the following well-known identity in the theory of Jordan pairs [15]:

$$\text{(1.4)} \qquad U_x y = \frac{1}{2}\big[x, [x, \bar{y}]\big] \quad \text{where} \quad \bar{y} := \frac{1}{2}\big[f_{\dot{\alpha}}, [f_{\dot{\alpha}}, y]\big] \quad \text{for all } x, y \in J,$$

where the U-operator U_x will be defined in (2.0). Also, we need the following easy consequence of sl_2-theory:

Lemma 1.5. *For $\sigma, \tau \in S$ and $\varepsilon = \pm 1$, we have*

$$\big[e_{\varepsilon\dot\alpha+\sigma}, [e_{\varepsilon\dot\alpha+\sigma}, e_{-\varepsilon\dot\alpha+\tau}]\big] \neq 0,$$

and hence, $\big[\mathcal{L}_{\varepsilon\dot\alpha+\sigma}, [\mathcal{L}_{\varepsilon\dot\alpha+\sigma}, \mathcal{L}_{-\varepsilon\dot\alpha+\tau}]\big] = \mathcal{L}_{\varepsilon\dot\alpha+2\sigma+\tau}$.

Proof. We only show the case $\varepsilon = 1$ since the case $\varepsilon = -1$ is done by the same manner. Take $\alpha := -\dot\alpha - \sigma$ and $\beta := -\dot\alpha + \tau$ in 1.1(iii). Then $[e_\alpha, e_\beta] \in \mathcal{L}_{-2\dot\alpha-\sigma+\tau} = (0)$ since A_1 is reduced. Also, we have $2\frac{(\beta,\alpha)}{(\alpha,\alpha)} = 2$, and hence by 1.1(iii), $\big[f_{-\dot\alpha-\sigma}, [f_{-\dot\alpha-\sigma}, e_{-\dot\alpha+\tau}]\big] \neq 0$. Since $\dim_{\mathbb{C}} \mathcal{L}_{\dot\alpha+\sigma} = 1$ (see 1.1(ii)), there exists some $0 \neq c \in \mathbb{C}$ such that $f_{-\dot\alpha-\sigma} = ce_{\dot\alpha+\sigma}$, and so $0 \neq \big[e_{\dot\alpha+\sigma}, [e_{\dot\alpha+\sigma}, e_{-\dot\alpha+\tau}]\big] \in \mathcal{L}_{\varepsilon\dot\alpha+2\sigma+\tau}$. □

For $\sigma \in S$, let $0 \neq x \in J_\sigma$. Since S is a semilattice, there exists $0 \neq y \in J_{-2\sigma}$. Then, by 1.4 and 1.5, we have $0 \neq \frac{1}{2}\big[x, [x, \bar y]\big] = U_x y \in J_{2\sigma-2\sigma} = J_0$. Since $\dim_{\mathbb{C}} J_{-2\sigma} = \dim_{\mathbb{C}} J_0 = 1$ and $1 \in J_0$, there exists $c \in \mathbb{C}$ such that $U_x(cy) = 1$. Hence, x is invertible (see 2.2(2)). Thus any nonzero element in J_σ for all $\sigma \in S$ is invertible. Consequently, we have obtained some necessary conditions of the coordinate algebra J, namely,

Theorem 1.6. *The core of an EALA of type A_1 is isomorphic to a central extension of the TKK algebra of a unital Λ-graded Jordan algebra $J = \oplus_{\sigma\in\Lambda} J_\sigma$ over $\mathbb{C}$ satisfying*

(T1) $\{\sigma \in \Lambda \mid J_\sigma \neq (0)\}$ *generates* Λ,
(T2) *all nonzero homogeneous elements are invertible,*
(T3) $\dim_{\mathbb{C}} J_\sigma \leq 1$ *for all* $\sigma \in \Lambda$. □

We will classify such Jordan algebras not only over $\mathbb{C}$ but over any field F of ch. $F \neq 2$ in later sections. By the argument above, we know that $1 \in J_0$ and that $\{\sigma \in \Lambda \mid J_\sigma \neq (0)\}$ is a semilattice in Λ. However, we do not need to assume the properties since such Jordan algebras already satisfy them (see 3.5).

§2 Review of Jordan algebras

Throughout F is a field of characteristic $\neq 2$. An algebra over F is a "linear" nonassociative algebra A defined as a vector space over F with an F-bilinear map $A \times A \longrightarrow A$, called *multiplication*. We assume that an algebra is unital in the sense that there exists $1 \in A$, called an *identity element*, such that $1x = x = x1$ for all $x \in A$. For an algebra A and $x, y, z \in A$ we define the *commutator* $[x, y] = xy - yx$ and the *associator* $(x, y, z) = (xy)z - x(yz)$.

An algebra J over F satisfying the following two identities is called a *(linear) Jordan algebra* over F: for all $x, y \in J$,

$$[x, y] = 0 \quad \text{(commutativity)} \quad \text{and} \quad (x, y, x^2) = 0 \quad (\textit{Jordan identity}).$$

Let us define the so-called *U-operator* for $x \in J$, i.e., $U_x : J \longrightarrow J$ by

$$U_x y = U_x(y) = 2x(xy) - x^2 y \quad \text{for all } y \in J. \tag{2.0}$$

The *plus algebra* A^+ *of an associative algebra* A over F is an example of a Jordan algebra: for $x, y \in A^+ = (A, \cdot)$, with a new multiplication $\cdot$ on A defined as $x \cdot y := \frac{1}{2}(xy + yx)$. For this example, the U-operator is given by $U_x y = xyx$. A Jordan algebra is called *special* if it is isomorphic to a subalgebra of the plus algebra of some associative algebra. A Jordan algebra is called *exceptional* if it is not special.

Remark 2.1. It is well known that Jordan algebras are *power associative*, i.e., the subalgebra generated by any element is associative (and commutative) (see e.g. [21] p.37, p.68).

An element x in a Jordan algebra J is called *invertible* if there exists $y \in J$ such that $xy = 1$ and $x^2 y = x$. In this case y is unique and is denoted by x^{-1}.

We denote the subset of invertible elements of an algebra A by $A^\times$.

2.2. ([21] p.303-4) *Let A be an associative algebra and J a Jordan algebra. Then:*

(1) *$A^\times = (A^+)^\times$, and for $x \in A^\times = (A^+)^\times$, x^{-1} in the associative algebra A and the Jordan algebra A^+ coincide.*

(2) *For $x \in J$, $x \in J^\times \iff U_x$ is invertible $\iff$ there exists $y \in J$ such that $U_x y = 1$. In these cases, we have $U_x^{-1} = U_{x^{-1}}$ and $y = x^{-2}$.*

(3) *For $x, y \in J$, $x, y \in J^\times \iff U_x y \in J^\times$. In particular, $x \in J^\times \iff x^n \in J^\times$ for all $n \in \mathbb{Z}$.*

(4) *For any $x \in J^\times$, the subalgebra of J generated by x and x^{-1} is associative (and commutative).* $\square$

We recall some basic notions for Jordan algebras.

Definition 2.3. Let J be a Jordan algebra. Then J is called

(i) a *Jordan domain* if $U_x y = 0$ implies $x = 0$ or $y = 0$ for all $x, y \in J$,

(ii) *nondegenerate* if $U_x = 0$ implies $x = 0$ for all $x \in J$,

(iii) *prime* if $U_I K = (0)$ implies $I = (0)$ or $K = (0)$ for all ideals I, K of J where $U_I K = \{\sum_{x,y} U_x y \mid x \in I,\ y \in K\}$,

(iv) *strongly prime* if J is nondegenerate and prime.

The nondegeneracy and primeness generalize the notion of a domain, namely,

Lemma 2.4. *A Jordan domain has no nilpotents and is strongly prime.*

Proof. Straightforward. $\square$

The *centre* of a Jordan algebra J is defined as

$$Z(J) = \{z \in J \mid (z, x, y) = 0 \text{ for all } x, y \in J\}.$$

We note that if $z \in Z(J)$ is invertible, then $z^{-1} \in Z(J)$. The following lemma is well-known (see [13] Corollary 3.4, p.12):

2.5. *Let A be a semiprime associative algebra and $Z(A)$ its centre. Then we have $Z(A) = Z(A^+)$.* $\square$

For a prime associative or Jordan algebra A, any $0 \neq x \in Z := Z(A)$ has no torsion element in A, i.e., $xy = 0$ for some $y \in A \Longrightarrow y = 0$ (see [9] Proposition 7.6.5, p.7.24). In particular, Z is an integral domain. Thus we can define the tensor algebra $\overline{A} = \overline{Z} \otimes_Z A$ over $\overline{Z}$ where $\overline{Z}$ is the field of fractions of Z, and call it the *central closure of* A. The following lemma is well-known (see e.g. [21] p.186):

2.6. *Let A be a prime associative or Jordan algebra and $Z = Z(A)$ its centre. Then we have:*

(i) *A embeds into $\overline{A}$ via $x \mapsto 1 \otimes x$ for all $x \in A$,*

(ii) *$\overline{A}$ is a central over $\overline{Z}$, i.e., $Z(\overline{A}) = \overline{Z}.1$,*

(iii) *A is an associative (resp. a Jordan) domain $\Longleftrightarrow$ $\overline{A}$ is an associative (resp. a Jordan) domain.* $\square$

We mention some well-known identities on A^+ for any associative algebra A. Define $x \circ y = xy + yx$ for $x, y \in A$ and let $(\cdot,\cdot,\cdot)^\circ$ be the associator of this circle product. Then the following identities which can be easily verified by expanding both sides hold: for all $x, y, z \in A$,

$$\big[x, [y, z]\big] = (y, x, z)^\circ, \tag{2.7}$$

$$[x, y]^2 = x \circ U_y x - U_x y^2 - U_y x^2, \tag{2.8}$$

$$U_{[x,y]} = U_{x \circ y} - 2U_x U_y - 2U_y U_x. \tag{2.9}$$

By Wedderburn's Structure Theorem, a finite dimensional associative domain is a division algebra. Also, the following is well-known (see [8] p.156 Theorem 2).

2.10. *A finite dimensional Jordan domain is a division algebra.* $\square$

Finally, there is a notion of degree for finite dimensional simple Jordan algebras (see e.g. [8], p.209), and the degree coincides with the generic degree for them (see [8], p.233). The following lemma seems to be known to the experts, but for the convenience of the reader we include a proof.

2.11. *Let J be a finite dimensional central special Jordan division algebra over F of degree r. Then:*

(a) $r \neq 2^m$ *for* $m \geq 1 \Longrightarrow \dim_F J = r^2$,

(b) $r = 3 \Longleftrightarrow \dim_F J = 9$.

Proof. (a): It is clear for $r = 1$, and so we assume that $r > 1$. If J is a finite dimensional central special Jordan division algebra over F of degree $r \neq 2$, then $J \cong D^+$ or $H(D, *)$ where D is a central associative division over the centre of degree r and $*$ is an involution of D (see Theorem 11 and Exercise 1 in [8] p.210). Thus, if $J \cong D^+$, then we have $\dim_F J = r^2$. If $*$ is of the second kind, we know $\dim_F H(D, *) = r^2$ (see [10] p.190). If $J \cong H(D, *)$ and $r \neq 2^m$ for $m \geq 1$, then there does not exist an involution of first kind on D. For, if one exists, then $D \cong D^{op}$ (the opposite algebra), and so the order of D in the Brauer group of F is 2. Since any prime factor of the degree of D divides the order (see Theorem 2.7.5 [10] p.61), the degree of D has to be a power of 2. This is a contradiction. Hence (a) has been shown.

(b): By (a), we get $r = 3 \Longrightarrow \dim_F J = 9$. Suppose that $\dim_F J = 9$. From the classification of the finite dimensional simple associative algebras with involution (see [10] p.190), we have $\dim_F J = r^2$ ($J \cong D^+$ or $*$ is of the second kind), $r(r+1)/2$ ($*$ is orthogonal) or $r(r-1)/2$ ($*$ is symplectic). Since $r(r+1)/2$ or $r(r-1)/2$ is never 9, we get $9 = r^2$, i.e., $r = 3$. $\square$

§3 General properties of Jordan tori

Whenever a class of algebras has a notion of invertibility, one can make the following definition:

Definition 3.1. Let G be a group. A G-graded algebra $T = \oplus_{g\in G} T_g$ over F satisfying

(T1) $\operatorname{supp} T := \{g \in G \mid T_g \neq (0)\}$ generates G,
(T2) all nonzero homogeneous elements are invertible,
(T3) $\dim_F T_g \leq 1$ for all $g \in G$,

is called a *G-torus*. Moreover, if

(St) T is strongly graded, i.e., $T_g T_h = T_{gh}$ for all $g, h \in G$,

then T is called a *G-torus of strong type*.

When $G = \Lambda$ is a free abelian group of rank n, T is called an *n-torus*, or simply a *torus*. If T is associative or Jordan, it is called an *associative torus* or a *Jordan torus*, respectively.

One can easily check that if $T = \oplus_{g\in G} T_g$ is a G-graded associative algebra satisfying (T3), then (T1) and (T2) are equivalent to (St). Thus the notions of a G-torus and a G-torus of strong type coincide for the class of associative algebras. Note for a G-torus $T = \oplus_{g\in G} T_g$ of strong type, we have $\dim_F T_g = 1$ for all $g \in G$ and $\operatorname{supp} T = G$. We give examples of associative tori and Jordan tori.

From now on, Λ denotes a free abelian group of rank n.

Example 3.2. Let $\{\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n\}$ be a basis of Λ. We give a Λ-grading to a quantum torus $F_{\boldsymbol{q}} = F_{\boldsymbol{q}}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ (see the definition in Introduction) in the following way: Define the degree of $t_{\boldsymbol{\alpha}} := t_1^{\alpha_1} \cdots t_n^{\alpha_n}$ for $\boldsymbol{\alpha} = \alpha_1\boldsymbol{\sigma}_1 + \cdots + \alpha_n\boldsymbol{\sigma}_n \in \Lambda$ to be $\boldsymbol{\alpha}$. Then this grading makes $F_{\boldsymbol{q}} = \oplus_{\boldsymbol{\alpha}\in\Lambda} Ft_{\boldsymbol{\alpha}}$ into a torus, and we call the grading a *toral Λ-grading of $F_{\boldsymbol{q}}$*. If one needs to specify a basis of Λ, we call it a *$\langle \boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n\rangle$-grading of $F_{\boldsymbol{q}}$*. Any associative torus is graded isomorphic to some $F_{\boldsymbol{q}}$ with some toral grading (see [5] and [6]). Also, any commutative associative torus is graded isomorphic to $F_{\boldsymbol{1}} = F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, the algebra of Laurent polynomials over F. One can check that the multiplication rule of $F_{\boldsymbol{q}}$ for $\boldsymbol{q} = (q_{ij})$ is the following: for $\boldsymbol{\beta} = \beta_1\boldsymbol{\sigma}_1 + \cdots + \beta_n\boldsymbol{\sigma}_n \in \Lambda$,

$$t_{\boldsymbol{\alpha}} t_{\boldsymbol{\beta}} = \prod_{i<j} q_{ij}^{\alpha_j \beta_i} t_{\boldsymbol{\alpha}+\boldsymbol{\beta}}. \tag{3.3}$$

Clearly, $F_{\boldsymbol{q}}^+$ with the same grading as $F_{\boldsymbol{q}}$ becomes a Jordan torus and the multiplication rule is the following:

$$\begin{aligned} t_{\boldsymbol{\alpha}} \cdot t_{\boldsymbol{\beta}} &= \frac{1}{2}\big(\prod_{i<j} q_{ij}^{\alpha_j\beta_i} + \prod_{i<j} q_{ij}^{\beta_j\alpha_i}\big)\, t_1^{\alpha_1+\beta_1}\cdots t_n^{\alpha_n+\beta_n} \\ &= \frac{1}{2}\prod_{i<j} q_{ij}^{\alpha_j\beta_i}\big(1+\prod_{i,j} q_{ij}^{\alpha_i\beta_j}\big)\, t_{\boldsymbol{\alpha}+\boldsymbol{\beta}}. \end{aligned} \tag{3.4}$$

We call the grading of $F_{\boldsymbol{q}}^+$ induced from a toral grading of $F_{\boldsymbol{q}}$ a *toral grading of* $F_{\boldsymbol{q}}^+$. Note that $\operatorname{supp} F_{\boldsymbol{q}}^+ = \Lambda$, and $F_{\boldsymbol{q}}^+$ is of strong type if and only if

$$\prod_{i,j} q_{ij}^{\alpha_i\beta_j} \neq -1 \quad \text{for all} \quad \boldsymbol{\alpha},\boldsymbol{\beta}\in\Lambda.$$

Unlike the situation for associative tori, we may have $\operatorname{supp} J \neq \Lambda$ for a Jordan torus J in general. We will give such examples in §4 and §5. We show that $\operatorname{supp} J$ cannot be any subset of Λ, namely,

Lemma 3.5. *Let $J = \oplus_{\boldsymbol{\alpha}\in\Lambda} J_{\boldsymbol{\alpha}}$ be a Jordan torus. Then $1 \in J_{\mathbf{0}}$ and* $\operatorname{supp} J$ *is a semilattice in Λ.*

Proof. In general, we have:

Claim. *Let $A = \oplus_{g\in G} A_g$ be a G-graded algebra. Then $1 \in A_e$ where e is the identity element of G.*

Proof. Let $1 = \sum_{g\in G} x_g \in A$. For any $u \in A_h$, $h \in G$, we have $u = 1u = \sum_{g\in G} x_g u \in A_h$ since 1 is a left identity element. Since G is a group, we have $x_e u = u$ (and $x_g u = 0$ if $g \neq e$). Thus x_e is a left identity element. Hence we have $1 = x_e 1 = x_e \in A_e$ since 1 is a right identity element. $\square$

By this claim, we have $1 \in J_{\mathbf{0}}$. In particular, $\mathbf{0} \in \operatorname{supp} J$. Let $0 \neq x \in J_{\boldsymbol{\alpha}}$ for $\boldsymbol{\alpha} \in \operatorname{supp} J$. Let $x^{-1} = \sum_{\boldsymbol{\beta}} y_{\boldsymbol{\beta}}$. Then $\sum_{\boldsymbol{\beta}} x y_{\boldsymbol{\beta}} = 1 \in J_{\mathbf{0}}$ and $\sum_{\boldsymbol{\beta}} x^2 y_{\boldsymbol{\beta}} = x \in J_{\boldsymbol{\alpha}}$ imply that $x y_{-\boldsymbol{\alpha}} = 1$ and $x^2 y_{-\boldsymbol{\alpha}} = x$ ($x y_{\boldsymbol{\gamma}} = 0$ and $x^2 y_{\boldsymbol{\gamma}} = 0$ for all $\boldsymbol{\gamma} \neq -\boldsymbol{\alpha}$). Thus, by the uniqueness of the inverse, we get $x^{-1} = y_{-\boldsymbol{\alpha}} \in J_{-\boldsymbol{\alpha}}$, and so $-\boldsymbol{\alpha} \in \operatorname{supp} J$. For any $\boldsymbol{\alpha},\boldsymbol{\beta} \in \operatorname{supp} J$, let $0 \neq u \in J_{\boldsymbol{\alpha}}$ and $0 \neq v \in J_{-\boldsymbol{\beta}}$. Then, by 2.2(3), $0 \neq U_u v = 2u(uv) - u^2 v \in J_{2\boldsymbol{\alpha}-\boldsymbol{\beta}}$, and so $2\boldsymbol{\alpha}-\boldsymbol{\beta} \in \operatorname{supp} J$. Since $\operatorname{supp} J$ generates Λ, $\operatorname{supp} J$ is a semilattice in Λ. $\square$

Remark. Any Jordan torus $J = \oplus_{\boldsymbol{\alpha}\in\Lambda} J_{\boldsymbol{\alpha}}$ satisfies $U_{J_{\boldsymbol{\alpha}}} J_{\boldsymbol{\beta}} = J_{2\boldsymbol{\alpha}+\boldsymbol{\beta}}$ for all $\boldsymbol{\alpha},\boldsymbol{\beta} \in \operatorname{supp} J$.

We recall that Λ is a totally ordered abelian group using the lexicographic order. Thus we have:

Lemma 3.6. (i) *A Jordan or an associative torus is a Jordan or an associative domain.*

(ii) *Any invertible element of a Jordan or an associative torus is homogeneous.*

Proof. Both statements are well-known for the associative case (see [11] p.95), and so we only show the Jordan case. Let $J = \oplus_{\boldsymbol{\alpha}\in\Lambda} J_{\boldsymbol{\alpha}}$ be a Jordan torus. For $0 \neq x, y \in J$, suppose that $U_x y = 0$. Let $x = x_{\boldsymbol{\alpha}_0} +$ (terms of higher degree) and $y = y_{\boldsymbol{\beta}_0} +$ (terms of higher degree). Then $U_{x_{\boldsymbol{\alpha}_0}} y_{\boldsymbol{\beta}_0}$ is the least homogeneous component of $U_x y$, and so $U_{x_{\boldsymbol{\alpha}_0}} y_{\boldsymbol{\beta}_0} = 0$. This is a contradiction since $x_{\boldsymbol{\alpha}_0}$ and $y_{\boldsymbol{\beta}_0}$ are invertible (see 2.2(3)). Hence (i) is settled.

For (ii), suppose that $x \in J$ is invertible but not homogeneous, i.e., $x = x_{\boldsymbol{\alpha}_0} +$ (terms of middle degree) $+\, x_{\boldsymbol{\alpha}_1}$ where $\boldsymbol{\alpha}_0$ is the minimum degree and $\boldsymbol{\alpha}_1$ is the maximum degree of x with $\boldsymbol{\alpha}_0 < \boldsymbol{\alpha}_1$. By 2.2(2), there exists $y \in J$ such that $U_x y = 1$. Let $y = y_{\boldsymbol{\beta}_0} +$ (terms of middle degree) $+\, y_{\boldsymbol{\beta}_1}$ where $\boldsymbol{\beta}_0$ is the minimum degree and $\boldsymbol{\beta}_1$ is the maximum degree of y (could be $\boldsymbol{\beta}_0 = \boldsymbol{\beta}_1$). Then $U_{x_{\boldsymbol{\alpha}_0}} y_{\boldsymbol{\beta}_0} \in J_{2\boldsymbol{\alpha}_0+\boldsymbol{\beta}_0}$ is the minimum degree and $U_{x_{\boldsymbol{\alpha}_1}} y_{\boldsymbol{\beta}_1} \in J_{2\boldsymbol{\alpha}_1+\boldsymbol{\beta}_1}$ is the maximum degree of $U_x y$, and $2\boldsymbol{\alpha}_0 + \boldsymbol{\beta}_0 < 2\boldsymbol{\alpha}_1 + \boldsymbol{\beta}_1$. This contradicts the fact that $U_x y = 1$ is homogeneous. $\square$

Corollary 3.7. *Let $T = \oplus_{\boldsymbol{\alpha}\in\Lambda} T_{\boldsymbol{\alpha}}$ be a Jordan or an associative torus. Suppose that $0 \neq x \in T$ and $x^m \in T_{\boldsymbol{\beta}}$ for some $\boldsymbol{\beta} \in \Lambda$. Then we have $\boldsymbol{\beta} \in m\Lambda$ and $x \in T_{\frac{1}{m}\boldsymbol{\beta}}$.*

Proof. By 3.6(i) and 2.4, we have $x^m \neq 0$. Since $x^m \in T_{\boldsymbol{\beta}}$, x^m is invertible, whence x is invertible (see 2.2(3)). Therefore, by 3.6(ii), we have $x \in T_{\boldsymbol{\gamma}}$ for some $\boldsymbol{\gamma} \in \Lambda$, and so $m\boldsymbol{\gamma} = \boldsymbol{\beta}$. $\square$

Let $T = \oplus_{\boldsymbol{\alpha}\in\Lambda} T_{\boldsymbol{\alpha}}$ be a Jordan or an associative torus. Then the centre $Z = Z(T)$ of T is a homogeneous subalgebra. Also, it is clear that Z is graded by the subgroup $\Gamma := \{\boldsymbol{\gamma} \in \Lambda \mid T_{\boldsymbol{\gamma}} \cap Z \neq (0)\}$ of Λ, and so $Z = \oplus_{\boldsymbol{\gamma}\in\Gamma} T_{\boldsymbol{\gamma}}$ is a commutative associative Γ-torus, which is the algebra of Laurent polynomials. We call this Γ the *central grading group of T*.

Lemma 3.8. *Let $T = \oplus_{\boldsymbol{\alpha}\in\Lambda} T_{\boldsymbol{\alpha}}$ be a Jordan or an associative torus over F with centre Z. Let K be a field extension of F. Then the scalar extension $T_K = K \otimes_F J$ is a Jordan or an associative torus over K with centre $K \otimes_F Z$, and the central grading groups of T and T_K coincide.*

Proof. Straightforward. $\square$

For a Jordan or an associative torus T, the central closure $\overline{T} = \overline{Z} \otimes_Z T$ over $\overline{Z}$ makes sense (see 3.6(i)).

Lemma 3.9. *Let $T = \oplus_{\boldsymbol{\alpha}\in\Lambda} T_{\boldsymbol{\alpha}}$ be a Jordan or an associative torus with its centre Z, Γ the central grading group of T and $\overline{T}$ the central closure. For $\overline{\boldsymbol{\alpha}} \in \Lambda/\Gamma$, let $T_{\overline{\boldsymbol{\alpha}}} := ZT_{\boldsymbol{\alpha}}$ and $\overline{T}_{\overline{\boldsymbol{\alpha}}} := \overline{Z} \otimes_Z ZT_{\boldsymbol{\alpha}}$. Then:*

(i) *$T_{\overline{\boldsymbol{\alpha}}} = T_{\overline{\boldsymbol{\alpha}'}}$ for all $\boldsymbol{\alpha}' \in \overline{\boldsymbol{\alpha}}$, and $T_{\overline{\boldsymbol{\alpha}}}$ is a free Z-module of rank 1 if $\boldsymbol{\alpha} \in \operatorname{supp} T$ and rank 0 otherwise.*

(ii) *$T = \oplus_{\overline{\boldsymbol{\alpha}}\in\Lambda/\Gamma} T_{\overline{\boldsymbol{\alpha}}}$, which is a free Z-module and a Λ/Γ-graded algebra over Z with $\operatorname{rank} T_{\overline{\boldsymbol{\alpha}}} \leq 1$ for all $\overline{\boldsymbol{\alpha}} \in \Lambda/\Gamma$.*

(iii) *$\overline{T} = \oplus_{\overline{\boldsymbol{\alpha}}\in\Lambda/\Gamma} \overline{T}_{\overline{\boldsymbol{\alpha}}}$, which is a Λ/Γ-torus over $\overline{Z}$ with $\dim_{\overline{Z}} \overline{T} = |\operatorname{supp} T/\Gamma|$.*

(iv) *The quotient group Λ/Γ cannot be a nontrivial cyclic group.*

Proof. (i) is trivial. For (ii), we note that for all $\boldsymbol{\gamma} \in \Gamma$, $T_{\boldsymbol{\alpha}+\boldsymbol{\gamma}} = T_{\boldsymbol{\gamma}} T_{\boldsymbol{\alpha}}$ since $T_{\boldsymbol{\gamma}} \subset Z$. Hence we have

$$T = \oplus_{\boldsymbol{\alpha}\in\Lambda} T_{\boldsymbol{\alpha}} = \oplus_{\overline{\boldsymbol{\alpha}}\in\Lambda/\Gamma} \left(\oplus_{\boldsymbol{\gamma}\in\Gamma} \ T_{\boldsymbol{\alpha}+\boldsymbol{\gamma}} \right) = \oplus_{\overline{\boldsymbol{\alpha}}\in\Lambda/\Gamma} ZT_{\boldsymbol{\alpha}} = \oplus_{\overline{\boldsymbol{\alpha}}\in\Lambda/\Gamma} T_{\overline{\boldsymbol{\alpha}}}.$$

The rest of statements follows from (i). (iii) follows from (ii). For (iv), suppose that Λ/Γ is a nontrivial cyclic group. One can easily check that any Jordan or associative G-torus for a cyclic group G is commutative and associative. Hence, by (iii), $\overline{T}$ is commutative and associative. By 2.6(i), T embeds into $\overline{T}$, and so $Z = T$ and $\Gamma = \Lambda$, i.e., Λ/Γ is the trivial group. Thus we get a contradiction. $\square$

We will start to classify Jordan tori in the next section. For this purpose we state Zelmanov's Prime Structure Theorem ([14] p.200) in a short form, designed for our needs. Namely, a strongly prime Jordan algebra $\mathcal{J}$ is one of the following three types: (The new terminology used below will be explained in the following sections.)

Hermitian type: $\mathcal{J}$ is special and $q_{48}(\mathcal{J}) \neq \{0\}$,

Clifford type: the central closure $\overline{\mathcal{J}}$ is a simple Jordan algebra over $\overline{Z}$ of a symmetric bilinear form,

Albert type: the central closure $\overline{\mathcal{J}}$ is an Albert algebra over $\overline{Z}$.

Since Jordan tori are strongly prime (see 3.6(i) and 2.4), the type of Jordan tori is defined as above.

§4 Hermitian type

We review the so-called *Zelmanov polynomial* q_{48}. Let $\mathcal{FSJ}(X)$ be a free special Jordan algebra on an infinite set X of variables over F. That is, $\mathcal{FSJ}(X)$ is the subalgebra of $\mathcal{FA}(X)^+$ generated by X where $\mathcal{FA}(X)$ is the free associative algebra on X. For $x, y, z, w \in X$, let

$$p_{16}(x,y,z,w) = \big[[D^2_{x,y}(z)^2, D_{x,y}(w)], D_{x,y}(w)\big] \in \mathcal{FA}(X)$$

where $D_{x,y}(z) = \big[[x,y],z\big]$. For 12 variables $x_i, y_i, z_i, w_i \in X$, $i = 1,2,3$, let

$$q_{48} = \big[[p_{16}(x_1,y_1,z_1,w_1), p_{16}(x_2,y_2,z_2,w_2)], p_{16}(x_3,y_3,z_3,w_3)\big] \in \mathcal{FA}(X).$$

By 2.7, we have $p_{16}, q_{48} \in \mathcal{FSJ}(X)$. Moreover, q_{48} is homogeneous in each variable, i.e., all monomials of q_{48}, the monomials not only of the associative product but also of the Jordan product, have the same partial degree in each variable. Note that the total degree in 12 variables is 48. For any Jordan algebra $\mathcal{J}$, we denote the evaluation of q_{48} on $\mathcal{J}$ by $q_{48}(\mathcal{J})$.

An ideal $\mathcal{I} \lhd \mathcal{FSJ}(X)$ is called *formal* if for all permutations σ of X,

$$p(x_1, \dots, x_n) \in \mathcal{I} \Longrightarrow p\big(\sigma(x_1), \dots, \sigma(x_n)\big) \in \mathcal{I}.$$

For a formal ideal $\mathcal{I}$ of $\mathcal{FSJ}(X)$ and any special Jordan algebra $\mathcal{J}$, it is well-known that the evaluation $\mathcal{I}(\mathcal{J})$ is an ideal of $\mathcal{J}$ (see [14], p.144). We define an *r-tad* $\{p_1 \cdots p_r\}$ for $p_1, \dots, p_n \in \mathcal{FSJ}(X)$ as

$$\{p_1 \cdots p_r\} = p_1 \cdots p_r + p_r \cdots p_1.$$

In particular, $\{p_1p_2p_3p_4\}$ is called a *tetrad.* A formal ideal $\mathcal{H} \lhd \mathcal{FSJ}(X)$ is called *hermitian* if it is closed under tetrads, i.e.,

$$\{\mathcal{HHHH}\} \subset \mathcal{H}.$$

An ideal $\mathcal{I} \lhd \mathcal{FSJ}(X)$ is called a *linearization-invariant T-ideal* if $\mathcal{I}$ contains all the linearizations of any $p \in \mathcal{I}$ and $T(\mathcal{I}) \subset \mathcal{I}$ for any algebra endomorphism T of

$\mathcal{FSJ}(X)$. Let Q_{48} be the linearization-invariant T-ideal of $\mathcal{FSJ}(X)$ generated by q_{48}. Then it is known that Q_{48} is an hermitian ideal (see [14] p.198).

Now, let $\mathcal{J}$ be any strongly prime special Jordan algebra with $q_{48}(\mathcal{J}) \neq \{0\}$. Then the evaluation $Q_{48}(\mathcal{J})$ is a nonzero ideal of $\mathcal{J}$. Since $\mathcal{J}$ is special, there exists an associative algebra $\mathcal{A}$ with involution $*$ such that $\mathcal{J} \subset H(\mathcal{A}, *) = \{a \in \mathcal{A} \mid a^* = a\}$. Let $\mathcal{P}$ be the associative subalgebra of $\mathcal{A}$ generated by $Q_{48}(\mathcal{J})$. By the Special Hermitian Structure Theorem [14] p.146, one has $H(\mathcal{P}, *) = Q_{48}(\mathcal{J})$ and $\mathcal{P}$ is $*$-prime.

Lemma 4.1. *Let J be a Jordan torus over F of Hermitian type. Then $J = H(P, *)$ for some $*$-prime associative algebra P and P is generated by J.*

Proof. By the observation above, we already know

$$H(P, *) = Q_{48}(J) \lhd J$$

for some $*$-prime associative algebra P and P is generated by $Q_{48}(J)$. Thus we only need to show $Q_{48}(J) = J$. Let B be a basis of J over F such that B consists of homogeneous elements in J. Recall that Q_{48} contains q_{48} and all the linearizations of q_{48}. If the evaluations of q_{48} and all the linearizations of q_{48} on B vanish, then we have $q_{48}(J) = \{0\}$, which contradicts the fact that J is of Hermitian type. Hence there exist elements $b_1, \ldots, b_m \in B$ and $q'_{48} \in Q_{48}$ where $q'_{48} = q_{48}$ or some linearization of q_{48} such that $q'_{48}(b_1, \ldots, b_m) \neq 0$. Since q'_{48} is homogeneous in each variable, $q'_{48}(b_1, \ldots, b_m)$ is a nonzero homogeneous element in J, and hence it is invertible in J. Thus the ideal $Q_{48}(J)$ contains an invertible element, and so we get $Q_{48}(J) = J$. $\square$

Definition 4.2. A Jordan torus J is called an *Hermitian torus* if $J = H(P, *)$ for some $*$-prime associative algebra P and P is generated by J.

We note {Hermitian tori} $\supset$ {Jordan tori of Hermitian type}, but the other inclusion does not hold, e.g. the algebra of Laurent polynomials $F_{\mathbf{1}} = F[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ is not of Hermitian type since $q_{48}(F_{\mathbf{1}}) = \{0\}$. However, $F_{\mathbf{1}} = H(F_{\mathbf{1}}, \mathrm{id})$ is clearly a Hermitian torus.

Example 4.3. (1) The Jordan torus $F_{\boldsymbol{q}}^+$ (see Example 3.2) is a Hermitian torus. In fact, let $P = F_{\boldsymbol{q}} \oplus F_{\boldsymbol{q}}^{op}$ be the associative algebra, where $F_{\boldsymbol{q}}^{op}$ is the opposite algebra of $F_{\boldsymbol{q}}$, and let $*$ be the exchange involution of P. Then $F_{\boldsymbol{q}}^+ \cong H(P, *)$ and one can easily check that P satisfies the conditions unless $\boldsymbol{q} = \mathbf{1}$.

(2) Let $\boldsymbol{\varepsilon} = (\varepsilon_{ij})$ be a quantum matrix such that $\varepsilon_{ij} = 1$ or -1. We call such an $\boldsymbol{\varepsilon}$ *elementary.* For the quantum torus $F_{\boldsymbol{\varepsilon}} = F_{\boldsymbol{\varepsilon}}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, there exists a unique involution $*$ on $F_{\boldsymbol{\varepsilon}}$ such that $t_i^* = t_i$ for all i. Thus the symmetric elements $J := H(F_{\boldsymbol{\varepsilon}}, *)$ form a Jordan algebra. Since $*$ is graded relative to a $\langle \boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n \rangle$-grading of $F_{\boldsymbol{\varepsilon}}$, $J = \oplus_{\boldsymbol{\alpha} \in \Lambda} (Ft_{\boldsymbol{\alpha}} \cap J)$ is a Λ-graded algebra and $\dim_F(Ft_{\boldsymbol{\alpha}} \cap J) \leq 1$. In general, the inverse of a symmetric element is also symmetric. Since $t_1, \dots, t_n \in J$, J generates $F_{\boldsymbol{\varepsilon}}$ and $\operatorname{supp} J$ generates Λ. Thus $J = H(F_{\boldsymbol{\varepsilon}}, *)$ is a Hermitian n-torus. If $\boldsymbol{q} \neq \mathbf{1}$, then $(t_i t_j)^* = -t_i t_j$ for some i, j, and so we have $Ft_i t_j \cap J = \{0\}$. Hence $\boldsymbol{\sigma}_i + \boldsymbol{\sigma}_j \notin \operatorname{supp} J$. Therefore, $\operatorname{supp} J = \Lambda$ if and only if $\boldsymbol{q} = \mathbf{1}$, i.e., $J = F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. In particular, J is never of strong type unless $\boldsymbol{q} = \mathbf{1}$. We call the grading of $H(F_{\boldsymbol{\varepsilon}}, *)$ induced from a toral grading of $F_{\boldsymbol{\varepsilon}}$ a *toral grading of* $H(F_{\boldsymbol{\varepsilon}}, *)$. (In [2] p.16, $H(F_{\boldsymbol{\varepsilon}}, *)$ is used to construct EALA's of type C.)

(3) Let E be a quadratic field extension of F. Let σ_E be the non-trivial Galois automorphism of E over F. Let $\boldsymbol{\xi} = (\xi_{ij})$ be a quantum matrix over E such that

$$\sigma_E(\xi_{ij})\xi_{ij} = 1 \quad (\Longleftrightarrow \sigma_E(\xi_{ij}) = \xi_{ji}) \quad \text{for all} \quad i, j. \tag{4.4}$$

For the quantum torus $E_{\boldsymbol{\xi}} = E_{\boldsymbol{\xi}}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ over E, there exists a unique σ_E-semilinear involution σ on $E_{\boldsymbol{\xi}}$ over F such that $\sigma(t_i) = t_i$ for all i. Thus the symmetric elements $H(E_{\boldsymbol{\xi}}, \sigma)$ form a Jordan algebra over F, and the Λ-grading induced from a toral grading of $E_{\boldsymbol{\xi}}$ makes $H(E_{\boldsymbol{\xi}}, \sigma)$ into a Λ-graded algebra. One can easily check that $H(E_{\boldsymbol{\xi}}, \sigma)$ is a Hermitian torus over F with $\operatorname{supp} H(E_{\boldsymbol{\xi}}, \sigma) = \Lambda$. We call the grading of $H(E_{\boldsymbol{\xi}}, \sigma)$ induced from a toral grading of $E_{\boldsymbol{\xi}}$ a *toral grading of* $H(E_{\boldsymbol{\xi}}, \sigma)$. Also, we will identify $E \otimes_F H(E_{\boldsymbol{\xi}}, \sigma)$ with $E_{\boldsymbol{\xi}}^+$ via $x \otimes t \leftrightarrow xt$ for $x \in E$ and $t \in H(E_{\boldsymbol{\xi}}, \sigma)$.

The following lemma is known for Jordan division algebras in [9] p.8.24. This is true for Jordan domains, and the proof is the same. But for the convenience of the reader, we prove it.

Lemma 4.5. *Let J be a Jordan domain satisfying $J = H(P, *)$ for an associative algebra P with involution $*$ such that P is generated by J. Suppose that there exists $v \in P$ such that $vv^* = 0$ and $v + v^*$ is invertible in J. Then $J \cong B^+$ for some associative algebra B.*

Proof. For any $y \in J = H(P, *)$ we claim that $v^*yv = v^*v = vyv^* = 0$. Clearly we have $v^*yv \in H(P, *) = J$. Also, $U_{v^*yv}1 = (v^*yv)^2 = (v^*yv) \cdot (v^*yv) = v^*yvv^*yv = 0$ since $vv^* = 0$. Hence $v^*yv = 0$ since J is a Jordan domain. In particular, $v^*v = 0$ for $y = 1$. By the same argument, we get $vyv^* = 0$, and so our claim is settled.

Now, since $v+v^*$ is invertible in J, there exists $z \in J$ such that $(v+v^*)z(v+v^*) = 1$. By the claim, we have $e + e^* = 1$ where $e := vzv$ and $ee^* = e^*e = 0$. Also, we have $e = e(e + e^*) = e^2$ and $e^* = e^*(e + e^*) = e^{*2}$. Thus e and e^* are supplementary orthogonal idempotents in P. By the claim, we have $e^*Je = eJe^* = \{0\}$. Since J generates P, we get $e^*Pe = ePe^* = \{0\}$. Therefore, for the associative algebra $B := ePe$, we have $P = B \oplus B^*$. Since $J = H(P, *)$, we obtain an isomorphism of Jordan algebras $f : J = \{b + b^* \mid b \in B\} \xrightarrow{\sim} B^+$. $\square$

We write $A \cong_\Lambda B$ if A and B are Λ-graded isomorphic. Also, we write $A \cong_\Lambda F_{\boldsymbol{q}}$, $F_{\boldsymbol{q}}^+$, $H(F_{\boldsymbol{\varepsilon}}, *)$ or $H(E_{\boldsymbol{\xi}}, \sigma)$, if A is Λ-graded isomorphic to one of them 'for some toral grading'.

Lemma 4.6. *Let $J = \oplus_{\boldsymbol{\alpha}\in\Lambda} J_{\boldsymbol{\alpha}}$ be a Jordan torus over F. If $J \cong B^+$ for some associative algebra B, then B is an associative torus and $J \cong_\Lambda F_{\boldsymbol{q}}^+$ for some $\boldsymbol{q}$.*

Proof. Let $f : J \xrightarrow{\sim} B^+$ be an isomorphism and $B_{\boldsymbol{\alpha}} := f(J_{\boldsymbol{\alpha}})$ for all $\boldsymbol{\alpha} \in \Lambda$. Then $B^+ = \oplus_{\boldsymbol{\alpha}\in\Lambda} B_{\boldsymbol{\alpha}}$ is a Jordan torus such that $J \cong_\Lambda B^+$. We show that $B = \oplus_{\boldsymbol{\alpha}\in\Lambda} B_{\boldsymbol{\alpha}}$ is an associative torus. Since $\operatorname{supp} B = \operatorname{supp} B^+$, we have $\Lambda = \langle \operatorname{supp} B^+ \rangle = \langle \operatorname{supp} B \rangle$. Also, all nonzero elements of $B_{\boldsymbol{\alpha}}$ are invertible in B^+, and so are they in B. Since $\dim_F B_{\boldsymbol{\alpha}} = \dim_F J_{\boldsymbol{\alpha}} \le 1$, we only need to show that $B_{\boldsymbol{\alpha}} B_{\boldsymbol{\beta}} \subset B_{\boldsymbol{\alpha}+\boldsymbol{\beta}}$ for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Lambda$. Note that we have $B_{\boldsymbol{\alpha}} \cdot B_{\boldsymbol{\beta}} = B_{\boldsymbol{\alpha}} \circ B_{\boldsymbol{\beta}} \subset B_{\boldsymbol{\alpha}+\boldsymbol{\beta}}$. If $B_{\boldsymbol{\alpha}} = (0)$ or $B_{\boldsymbol{\beta}} = (0)$, we have nothing to prove. Otherwise, for $0 \ne x \in B_{\boldsymbol{\alpha}}$ and $0 \ne y \in B_{\boldsymbol{\beta}}$, xy and yx are invertible in B and so are they in B^+. Hence, by 3.6(ii), $xy \in B_{\boldsymbol{\gamma}}$ and $yx \in B_{\boldsymbol{\delta}}$ for some $\boldsymbol{\gamma}, \boldsymbol{\delta} \in \Lambda$. If $x \circ y = xy + yx \ne 0$, then $0 \ne x \circ y \in B_{\boldsymbol{\alpha}+\boldsymbol{\beta}} \cap (B_{\boldsymbol{\gamma}} + B_{\boldsymbol{\delta}})$, which forces $\boldsymbol{\alpha} + \boldsymbol{\beta} = \boldsymbol{\gamma} = \boldsymbol{\delta}$. So we get $xy \in B_{\boldsymbol{\alpha}+\boldsymbol{\beta}}$. If $x \circ y = xy + yx = 0$, i.e., $yx = -xy$, then we have $[x^2, y^2] = 0$, and so $x^2 \cdot y^2 = x^2y^2$. Thus we get

$$0 \ne (xy)^2 = (xy) \cdot (xy) = xyxy = -x^2y^2 = -x^2 \cdot y^2 \in B_{2\boldsymbol{\gamma}} \cap B_{2\boldsymbol{\alpha}+2\boldsymbol{\beta}}.$$

Hence $\boldsymbol{\gamma} = \boldsymbol{\alpha} + \boldsymbol{\beta}$ and we obtain $xy \in B_{\boldsymbol{\alpha}+\boldsymbol{\beta}}$. Therefore, $B = \oplus_{\boldsymbol{\alpha}\in\Lambda} B_{\boldsymbol{\alpha}}$ is an associative torus, and so $B \cong_\Lambda F_{\boldsymbol{q}}$ for some $\boldsymbol{q}$ and we get $J \cong_\Lambda F_{\boldsymbol{q}}^+$. $\square$

Since Jordan tori are Jordan domains, one gets (a) of the following by 4.5 and 4.6:

Proposition 4.7. *Let $J = \oplus_{\boldsymbol{\alpha}\in\Lambda} J_{\boldsymbol{\alpha}}$ be a Jordan torus over F satisfying $J = H(P, *)$ for an associative algebra P with involution $*$ such that P is generated by J.*

(a) *Suppose that there exists $v \in P$ such that $vv^* = 0$ and $v + v^*$ is invertible in J. Then $J \cong_\Lambda F_{\boldsymbol{q}}^+$ for some $\boldsymbol{q}$.*

(b) *Suppose that there exist an invertible element* $u \in P$ *so that* $u^* = -u$ *and* $0 \neq y \in J_{\boldsymbol{\gamma}}$ *for some* $\boldsymbol{\gamma} \in \Lambda$ *such that the following three conditions hold:*

$$(i) \quad u^2 \in J_{2\boldsymbol{\gamma}}, \quad (ii) \quad uy^{-1}u \in J_{\boldsymbol{\gamma}}, \quad (iii) \quad [u, y] \in J_{2\boldsymbol{\gamma}}.$$

Then $J \cong_\Lambda F_{\boldsymbol{q}}^+$ *or* $E \otimes_F J \cong_\Lambda E_{\boldsymbol{q}}^+$ *for some* $\boldsymbol{q}$ *where* E *is a quadratic field extension of* F.

Proof. We only need to show (b). By $\dim_F J_{\boldsymbol{\alpha}} \leq 1$ for all $\boldsymbol{\alpha} \in \Lambda$, there exist $a, b, c \in F$, $a, b \neq 0$, such that

$$u^2 = ay^2, \tag{1}$$

$$buy^{-1}u = y \quad (\Longleftrightarrow \quad yu^{-1} = buy^{-1}), \tag{2}$$

$$[u, y] = uy - yu = cu^2. \tag{3}$$

By (3), we have $uyu^{-1} - y = cu$. By (2), we have $bu^2y^{-1} - y = cu$. By (1), we have $aby^2y^{-1} - y = cu$. Hence we get $cu = aby - y \in J$. Since $u \notin J$, we obtain $c = 0$ and hence

$$uy = yu. \tag{4}$$

Let $v := u + \sqrt{a}\, y$ if $\sqrt{a} \in F$. Otherwise, let $E := F(\sqrt{a})$, $J_E := E \otimes_F J$, $P_E := E \otimes_F P$, $* := \mathrm{id} \otimes *$ and $v := 1 \otimes u + \sqrt{a} \otimes y = u + \sqrt{a}\, y$. Then since $u^* = -u$, we have

$$vv^* = \begin{cases} (u + \sqrt{a}\, y)(-u + \sqrt{a}\, y) = -u^2 + ay^2 = 0 \\ (1 \otimes u + \sqrt{a} \otimes y)(-1 \otimes u + \sqrt{a} \otimes y) = 1 \otimes (-u^2 + ay^2) = 0, \end{cases}$$

by (4) and (1). Also, $v + v^* = 2\sqrt{a}\, y$ or $2\sqrt{a} \otimes y$ is invertible in J or in J_E. Thus, by (a), we get $J \cong_\Lambda F_{\boldsymbol{q}}^+$ if $\sqrt{a} \in F$. If $\sqrt{a} \notin F$, then we can apply (a) for the Jordan torus $J_E = H(P_E, *)$ over E, and obtain $J_E \cong_\Lambda E_{\boldsymbol{q}}^+$. □

For the next proposition, we need the following fact:

4.8. *If* A *and* B *are associative algebras,* B *has no zero-divisors and* $f : A^+ \longrightarrow B^+$ *is a homomorphism of Jordan algebras, then* $f : A \longrightarrow B$ *is either a homomorphism or anti-homomorphism of associative algebras* ([9] Theorem 1.1.7, p.1.4). □

Proposition 4.9. *Let $E \supset F$ be fields with $[E : F] = 2$, and σ_E is the nontrivial Galois automorphism of E over F. Let $\boldsymbol{q} \in M_n(E)$ be a quantum matrix over E.*

(1) *Suppose that τ is a σ_E-semilinear involution of $E_{\boldsymbol{q}}^+ = E_{\boldsymbol{q}}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]^+$ over F such that $\tau(t_i) = t_i$ for all i. Then:*

(i) *$\boldsymbol{q} \in M_n(F)$ and τ is an automorphism of $E_{\boldsymbol{q}}$ over F or*

(ii) *$\boldsymbol{q} = \boldsymbol{\xi}$ and τ is an involution of $E_{\boldsymbol{\xi}}$ over F, where $\boldsymbol{\xi}$ satisfies* (4.4). *In particular, $\tau = \sigma$ where σ is defined in* 4.3(3).

(2) *Let J be a Jordan torus over F. Suppose that $J_E = E \otimes_F J \cong_\Lambda E_{\boldsymbol{q}}^+$. Then:*

(i) *$\boldsymbol{q} \in M_n(F)$ and $J \cong_\Lambda F_{\boldsymbol{q}}^+$ or*

(ii) *$J \cong_\Lambda H(E_{\boldsymbol{\xi}}, \sigma)$, which is a Hermitian torus defined in* 4.3(3). *In this case we can identify $E \otimes_F H(E_{\boldsymbol{\xi}}, \sigma) = E_{\boldsymbol{\xi}}^+$ so that $\sigma_E \otimes \mathrm{id} = \sigma$.*

Proof. (1): Since $E_{\boldsymbol{q}}$ has no zero-divisors, τ is an order 2 automorphism or an involution of the associative F-algebra $E_{\boldsymbol{q}}$ (see 4.8). If τ is an automorphism, then

$$t_j t_i - q_{ij} t_i t_j = 0 = \tau(t_j t_i - q_{ij} t_i t_j) = t_j t_i - \sigma_E(q_{ij}) t_i t_j,$$

which forces $\sigma_E(q_{ij}) = q_{ij}$, and so $q_{ij} \in F$ for all i, j, i.e., $\boldsymbol{q} \in M_n(F)$.

If τ is an involution, then

$$0 = \tau(t_j t_i - q_{ij} t_i t_j) = t_i t_j - \sigma_E(q_{ij}) t_j t_i = t_i t_j - \sigma_E(q_{ij}) q_{ij} t_i t_j.$$

Hence we get $\sigma_E(q_{ij}) q_{ij} = 1$ for all i, j and obtain $\boldsymbol{q} = \boldsymbol{\xi}$. In particular, $\tau = \sigma$ since σ is a unique σ_E-semilinear involution of $E_{\boldsymbol{\xi}}$ such that $\sigma(t_i) = t_i$ for all i.

(2): Let $\tau := \sigma_E \otimes \mathrm{id}$, which is a σ_E-semilinear involution of $J_E = E \otimes_F J$. Identifying J with $1 \otimes J$, we have $J = H(J_E, \tau)$, the set of fixed points by τ. Also, we identify J_E with $E_{\boldsymbol{q}}^+$, and so $J = H(E_{\boldsymbol{q}}^+, \tau)$. Since $\operatorname{supp} J = \operatorname{supp} J_E = \operatorname{supp} E_{\boldsymbol{q}}^+ = \Lambda$, one can choose $t_1, \dots, t_n \in E_{\boldsymbol{q}}$ such that $E_{\boldsymbol{q}} = E_{\boldsymbol{q}}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ and $\tau(t_i) = t_i$ for all i. Thus one can apply (1). For the case (i), one gets $H(E_{\boldsymbol{q}}, \tau) = F_{\boldsymbol{q}}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, and so $J = F_{\boldsymbol{q}}^+$. For the case (ii), one obtains $\boldsymbol{q} = \boldsymbol{\xi}$, $\tau = \sigma$ and $J = H(E_{\boldsymbol{\xi}}, \sigma)$. □

We note the following fact about semilattices:

4.10. *Let S be a semilattice in Λ. Then there exists a basis $\{\boldsymbol{\sigma}_1, \cdots, \boldsymbol{\sigma}_n\}$ of Λ such that each $\boldsymbol{\sigma}_i \in S$* ([1] p.24 Proposition 1.11). □

The reader is reminded that any Jordan division algebra of Hermitian type is isomorphic to A^+ or $H(B, *)$ for some associative division algebra A or some associative division algebra B with involution $*$. We are now ready to prove an analogous result for Hermitian tori.

Theorem 4.11. *Any Hermitian torus over F is graded isomorphic to one of the three tori $F_{\boldsymbol{q}}^+$, $H(F_{\boldsymbol{\varepsilon}}, *)$ or $H(E_{\boldsymbol{\xi}}, \sigma)$ for some toral grading, as described in 4.3, and conversely, these three tori are all Hermitian tori.*

Proof. We only need to show the first statement. Let $J = \oplus_{\boldsymbol{\alpha}\in\Lambda} J_{\boldsymbol{\alpha}}$ be a Hermitian torus over F, i.e., $J = H(P, *)$ for some $*$-prime associative algebra P which is generated by J. Let $\langle \boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n\rangle$ be a basis of Λ such that each $\boldsymbol{\sigma}_i \in \operatorname{supp} J$ (see 4.10), and let $0 \neq x_i \in J_{\boldsymbol{\sigma}_i}$, $i = 1, \dots, n$. We first consider the case where the following two conditions hold:

(A) for all $1 \leq i, j \leq n$, $[x_i, x_j] = 0$ or $x_i \circ x_j = 0$, i.e., $x_j x_i = \pm x_i x_j$ for all $1 \leq i, j \leq n$,

(B) J is generated by r-tads $\{x_{i_1}^{\varepsilon_1} \cdots x_{i_r}^{\varepsilon_r}\}$ where $r > 0$, $i_1, \dots, i_r \in \{1, \dots, n\}$ and $\varepsilon_k = \pm 1$.

We will show $J \cong_\Lambda H(F_{\boldsymbol{\varepsilon}}, *)$ in this case. Since J generates P and every r-tad is generated by 1-tads $2x_1^{\pm 1}, \dots, 2x_n^{\pm 1}$ as an associative algebra, P is generated by $x_1^{\pm 1}, \dots, x_n^{\pm 1}$. Thus, by (A), there exist an elementary quantum matrix $\boldsymbol{\varepsilon}$ and an epimorphism φ from $F_{\boldsymbol{\varepsilon}} = F_{\boldsymbol{\varepsilon}}[t_1^{\pm}, \dots, t_n^{\pm}]$ onto P such that $\varphi(t_i) = x_i$ for $i = 1, \dots, n$. We give $F_{\boldsymbol{\varepsilon}}$ the $\langle \boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n\rangle$-grading, and show injectivity of φ. Suppose that $\varphi(t) = 0$ for $t = \sum_{\boldsymbol{\alpha}\in\Lambda} a_{\boldsymbol{\alpha}} t_{\boldsymbol{\alpha}}$ where $a_{\boldsymbol{\alpha}} \in F$, $t_{\boldsymbol{\alpha}} = t_1^{\alpha_1} \cdots t_n^{\alpha_n}$ and $\boldsymbol{\alpha} = \alpha_1\boldsymbol{\sigma}_1 + \cdots + \alpha_n\boldsymbol{\sigma}_n \in \Lambda$. So we have

$$\sum_{\boldsymbol{\alpha}\in\Lambda} a_{\boldsymbol{\alpha}} x_{\boldsymbol{\alpha}} = 0, \tag{1}$$

where $x_{\boldsymbol{\alpha}} := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Note that all $x_{\boldsymbol{\alpha}}$ are invertible in P since x_i is invertible in J and in P. We need to show that all $a_{\boldsymbol{\alpha}} = 0$. Put

$$M := \{\boldsymbol{\alpha} \in \Lambda \mid x_{\boldsymbol{\alpha}} \in J\} \quad \text{and} \quad N := \{\boldsymbol{\beta} \in \Lambda \mid x_{\boldsymbol{\beta}} \notin J\}.$$

Then we have $\sum_{\boldsymbol{\alpha}\in M} a_{\boldsymbol{\alpha}} x_{\boldsymbol{\alpha}} = 0$ and $\sum_{\boldsymbol{\beta}\in N} a_{\boldsymbol{\beta}} x_{\boldsymbol{\beta}} = 0$ since $x_{\boldsymbol{\alpha}} \in J$ are symmetric and $x_{\boldsymbol{\beta}} \notin J$ are skew relative to $*$ (note ch. $F \neq 2$).

Claim 1. *Assuming only (A), we have $x_{\boldsymbol{\alpha}} \in J \Longrightarrow x_{\boldsymbol{\alpha}} \in J_{\boldsymbol{\alpha}}$.*

Proof. By (A), the subalgebra of A generated by $\{x_i^2\}_{i=1}^n$ is commutative, and so the Jordan product and the associative product coincide in the subalgebra. Therefore,

$$x_{\boldsymbol{\alpha}}^2 = \pm x_1^{2\alpha_1} \cdots x_n^{2\alpha_n} = \pm(\cdot\cdot(x_1^{2\alpha_1} \cdot x_2^{2\alpha_2}) \cdot \cdots \cdot x_n^{2\alpha_n}) \in J_{2\alpha_1\boldsymbol{\sigma}_1 + \cdots + 2\alpha_n\boldsymbol{\sigma}_n} = J_{2\boldsymbol{\alpha}}.$$

Hence, by 3.7, we get $x_{\boldsymbol{\alpha}} \in J_{\boldsymbol{\alpha}}$. $\square$

By Claim 1, we obtain $a_{\boldsymbol{\alpha}} = 0$ for all $\boldsymbol{\alpha} \in M$. If $N = \emptyset$, we are done. Otherwise we pick any $\boldsymbol{\beta}_0 \in N$. Multiply (1) by $x_{\boldsymbol{\beta}_0}$. Then, by (A), we have

$$\sum_{\boldsymbol{\beta} \in N} \pm a_{\boldsymbol{\beta}} x_{\boldsymbol{\beta}+\boldsymbol{\beta}_0} = 0.$$

Applying the same argument for this equation instead of (1), we have

$$\sum_{\boldsymbol{\beta} \in M_1} \pm a_{\boldsymbol{\beta}} x_{\boldsymbol{\beta}+\boldsymbol{\beta}_0} = 0 \quad \text{and} \quad \sum_{\boldsymbol{\gamma} \in N_1} \pm a_{\boldsymbol{\gamma}} x_{\boldsymbol{\gamma}+\boldsymbol{\beta}_0} = 0,$$

where $M_1 := \{\boldsymbol{\beta} \in N \mid x_{\boldsymbol{\beta}+\boldsymbol{\beta}_0} \in J\}$ and $N_1 := \{\boldsymbol{\gamma} \in N \mid x_{\boldsymbol{\gamma}+\boldsymbol{\beta}_0} \notin J\}$. By Claim 1, we get $a_{\boldsymbol{\beta}} x_{\boldsymbol{\beta}+\boldsymbol{\beta}_0} \in J_{\boldsymbol{\beta}+\boldsymbol{\beta}_0}$, and hence $a_{\boldsymbol{\beta}} = 0$ for all $\boldsymbol{\beta} \in M_1$. Since $N = M_1 \sqcup N_1$ (disjoint union) and $\boldsymbol{\beta}_0 \in M_1$ (because $x_{2\boldsymbol{\beta}_0} \in J$), we have $N_1 \subsetneq N$. If $N_1 = \emptyset$, we are done. Otherwise, repeating this method for the finite set N_1, we get some $r > 1$ such that $N_r = \emptyset$ and $a_{\boldsymbol{\beta}} = 0$ for all $\boldsymbol{\beta} \in M_2 \sqcup \cdots \sqcup M_r = N_1$. Consequently, we obtain $a_{\boldsymbol{\alpha}} = 0$ for all $\boldsymbol{\alpha} \in \Lambda$. Thus $t = 0$ and φ is injective.

We have shown that φ is an isomorphism. Further, P is graded with $P_{\boldsymbol{\sigma}_i} = J_{\boldsymbol{\sigma}_i}$ and φ is a graded isomorphism. Also, through this isomorphism, we get an involution $*$ of $F_{\boldsymbol{\varepsilon}}$ such that $t_i^* = t_i$ for $i = 1, \ldots, n$. Therefore, we obtain $J \cong_\Lambda H(F_{\boldsymbol{\varepsilon}}, *)$.

We consider the second case: the negation of (A), i.e.,

$$\text{there exist some } i, j \text{ such that } u := [x_i, x_j] \neq 0 \text{ and } x_i \circ x_j \neq 0.$$

We divide the case into two subcases:

$$\text{(I)} \quad u^2 = 0 \quad \text{and} \quad \text{(II)} \quad u^2 \neq 0.$$

Note that $u^* = -u$.

(I): We have $uu^* = -u^2 = 0$. We need the following claim which can be proven in the same manner as in the classification of Jordan division algebras (see [9] p.8.25).

Claim 2. *There exists $y \in J$ such that for $v = yu$, $v + v^* \neq 0$.*

Proof. Otherwise, for all $y \in J$, we have $v + v^* = 0$ for $v = yu$, i.e., $yu = -u^* y^* = uy$. So for all $w \in P$, we have $(uy)(uw) = u^2 yw = 0$, and hence we have $(uJ)(uP) = \{0\}$. Since P is generated by J, we get $(uP)^2 = \{0\}$. Then we have $(PuP)^2 =$

$PuPPuP = PuPuP = P(uP)^2 = \{0\}$. Moreover, $(PuP)^* = Pu^*P = PuP$, and so PuP is a nonzero $*$-ideal. This contradicts the fact that P is $*$-prime ($*$-semiprime is enough!). $\square$

Let $y \in J$ be such an element as in Claim 2 so that

$$v + v^* = yu - uy = [y, u] \neq 0.$$

Decompose y into nonzero distinct homogeneous elements, namely, $y = \sum_{\boldsymbol{\alpha}} y_{\boldsymbol{\alpha}}$. Let $v_{\boldsymbol{\alpha}} := y_{\boldsymbol{\alpha}} u$ for all $y_{\boldsymbol{\alpha}}$. Suppose that $v_{\boldsymbol{\alpha}} + v_{\boldsymbol{\alpha}}^* = 0$ for all $v_{\boldsymbol{\alpha}}$. Then we have $[y_{\boldsymbol{\alpha}}, u] = y_{\boldsymbol{\alpha}} u - u y_{\boldsymbol{\alpha}} = v_{\boldsymbol{\alpha}} + v_{\boldsymbol{\alpha}}^* = 0$ for all $y_{\boldsymbol{\alpha}}$. Therefore,

$$v + v^* = [y, u] = \big[\sum_{\boldsymbol{\alpha}} y_{\boldsymbol{\alpha}}, u\big] = \sum_{\boldsymbol{\alpha}} [y_{\boldsymbol{\alpha}}, u] = 0,$$

which contradicts our choice of y. Hence there exists some $v_{\boldsymbol{\alpha}}$ such that $v_{\boldsymbol{\alpha}} + v_{\boldsymbol{\alpha}}^* \neq 0$. By 2.7, we have

$$0 \neq v_{\boldsymbol{\alpha}} + v_{\boldsymbol{\alpha}}^* = [y_{\boldsymbol{\alpha}}, u] = \big[y_{\boldsymbol{\alpha}}, [x_i, x_j]\big] = (x_i, y_{\boldsymbol{\alpha}}, x_j)^\circ \in J_{\boldsymbol{\sigma}_i + \boldsymbol{\alpha} + \boldsymbol{\sigma}_j}.$$

Hence $v_{\boldsymbol{\alpha}} + v_{\boldsymbol{\alpha}}^*$ is invertible in J. Also, we have $v_{\boldsymbol{\alpha}} v_{\boldsymbol{\alpha}}^* = y_{\boldsymbol{\alpha}} u u^* y_{\boldsymbol{\alpha}}^* = -y_{\boldsymbol{\alpha}} u^2 y_{\boldsymbol{\alpha}} = 0$. Therefore, by 4.7(a), we get $J \cong_\Lambda F_{\boldsymbol{q}}^+$ for some $\boldsymbol{q}$.

(II): Let $0 \neq y := x_i \circ x_j \in J_{\boldsymbol{\gamma}}$ where $\boldsymbol{\gamma} := \boldsymbol{\sigma}_i + \boldsymbol{\sigma}_j$. We show that these u and y satisfy the three conditions in 4.7(b). By 2.8, we have

$$0 \neq u^2 = [x_i, x_j]^2 = x_i \circ U_{x_j} x_i - U_{x_i} x_j^2 - U_{x_j} x_i^2 \in J_{2\boldsymbol{\gamma}}.$$

Hence u^2 is invertible in J and hence in P. Thus u is invertible in P and $u^2 \in J_{2\boldsymbol{\gamma}}$. By 2.9, we have

$$u y^{-1} u = U_u y^{-1} = U_{[x_i, x_j]} y^{-1} = (U_{x_i \circ x_j} - 2U_{x_i} U_{x_j} - 2U_{x_j} U_{x_i}) y^{-1} \in J_{\boldsymbol{\gamma}}$$

since $y^{-1} \in J_{-\boldsymbol{\gamma}}$. By 2.7, we have

$$[u, y] = \big[[x_i, x_j], x_i \circ x_j\big] = -(x_i, x_i \circ x_j, x_j)^\circ \in J_{2\boldsymbol{\gamma}}.$$

Thus u and y satisfy the conditions in 4.7(b), and we get $J \cong_\Lambda F_{\boldsymbol{q}}^+$ or $J_E \cong_\Lambda E_{\boldsymbol{q}}^+$. Then, by 4.9(2), we obtain $J \cong_\Lambda F_{\boldsymbol{q}}^+$ or $H(E_{\boldsymbol{\xi}}, \sigma)$.

We consider the final case: (A) with the negation of (B), i,e.,

assuming the relation (A), J is not generated by r-tads $\{x_{i_1}^{\varepsilon_1} \cdots x_{i_r}^{\varepsilon_r}\}$.

By our assumption, there exist $\boldsymbol{\gamma} \in \Lambda$ and $0 \neq y \in J_{\boldsymbol{\gamma}}$ such that y is not generated by r-tads $\{x_{i_1}^{\varepsilon_1} \cdots x_{i_r}^{\varepsilon_r}\}$. Let $\boldsymbol{\gamma} = \gamma_1 \boldsymbol{\sigma}_1 + \cdots + \gamma_n \boldsymbol{\sigma}_n$.

Claim 3. $u := x_1^{\gamma_1} \cdots x_n^{\gamma_n} \notin J$.

Proof. Otherwise we have $2u = \{x_1^{\gamma_1} \cdots x_n^{\gamma_n}\}$, and by Claim 1, $u \in J_{\boldsymbol{\gamma}}$. So we get $y = au$ for some $a \in F$ by $\dim_F J_{\boldsymbol{\gamma}} \leq 1$, i.e., $y = \frac{1}{2}a\{x_1^{\gamma_1} \cdots x_n^{\gamma_n}\}$, which contradicts our setting of y. $\square$

Now, we show that these u and $y \in J_{\boldsymbol{\gamma}}$ satisfy the conditions in 4.7(b). Observe first that $u^* = \pm u$ by (A) and hence $u^* = -u$ by Claim 3. Next, by definition, u is clearly invertible, and by (A), we have $u^2 = \pm x_1^{2\gamma_1} \cdots x_n^{2\gamma_n} \in J$. Hence $u^2 \in J_{2\boldsymbol{\gamma}}$ by Claim 1. Secondly we have

$$\begin{aligned} U_u y^{-1} = u y^{-1} u &= x_1^{\gamma_1} \cdots x_n^{\gamma_n} y^{-1} x_1^{\gamma_1} \cdots x_n^{\gamma_n} = \pm x_1^{\gamma_1} \cdots x_n^{\gamma_n} y^{-1} x_n^{\gamma_n} \cdots x_1^{\gamma_1} \\ &= \pm U_{x_1^{\gamma_1}} \cdots U_{x_n^{\gamma_n}} y^{-1} \in J_{\boldsymbol{\gamma}}. \end{aligned}$$

In particular, we get a formula for u:

$$U_u = \pm U_{x_1^{\gamma_1}} \cdots U_{x_n^{\gamma_n}}. \tag{2}$$

Thirdly, since $u^* = -u$ and $[u, y]^* = [y^*, u^*] = -[y, u] = [u, y]$, we have $[u, y] \in J$. Also, we have by 2.8 and (2),

$$[u, y]^2 = y \circ U_u y - U_y u^2 - U_u y^2 \in J_{4\boldsymbol{\gamma}}.$$

Hence, by 3.7, we get $[u, y] \in J_{2\boldsymbol{\gamma}}$. Thus, by 4.7(b) and 4.9(2), we get $J \cong_\Lambda F_{\boldsymbol{q}}^+$ or $H(E_{\boldsymbol{\xi}}, \sigma)$. $\square$

§5 Clifford type

Definition 5.1. A Jordan torus of Clifford type, i.e., the central closure is a Jordan algebra of a symmetric bilinear form, is called a *Clifford torus*.

Example 5.2. Let $\Lambda = \Lambda_n$ be a free abelian group of rank $n \geq 2$. Let $2 \leq m \leq n$ and let Λ_m and Λ_{n-m} be subgroups of Λ of rank m and $n-m$, respectively, such that $\Lambda = \Lambda_m \oplus \Lambda_{n-m}$. Let $S^{(m)}$ be a semilattice in Λ_m and $\{\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_m\}$ a basis of Λ_m. Extend $\{\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_m\}$ to a basis $\{\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_m, \boldsymbol{\sigma}_{m+1}, \dots, \boldsymbol{\sigma}_n\}$ of Λ. Let

$$\Gamma := 2\Lambda_m \oplus \Lambda_{n-m} \quad \text{and} \quad Z := F[z_1^{\pm 1}, \dots, z_n^{\pm 1}] \text{ with a toral } \Gamma\text{-grading, i.e.,}$$

$Z = \oplus_{\boldsymbol{\gamma} \in \Gamma} F z_{\boldsymbol{\gamma}}$ where

$$z_{\boldsymbol{\gamma}} = z_1^{\gamma_1} \cdots z_n^{\gamma_n} \text{ for } \boldsymbol{\gamma} = 2\gamma_1 \boldsymbol{\sigma}_1 + \cdots + 2\gamma_m \boldsymbol{\sigma}_m + \gamma_{m+1} \boldsymbol{\sigma}_{m+1} + \cdots + \gamma_n \boldsymbol{\sigma}_n.$$

We fix

$$\text{a representative set } I \text{ of } (S^{(m)}/2\Lambda_m) \setminus \{\overline{\mathbf{0}}\}.$$

Let V be a free Z-module with basis $\{t_{\boldsymbol{\epsilon}}\}_{\boldsymbol{\epsilon}\in I}$. Define a Z-bilinear form $f : V\times V \longrightarrow Z$ by

$$(*) \qquad f(t_{\boldsymbol{\epsilon}}, t_{\boldsymbol{\eta}}) = \begin{cases} a_{\boldsymbol{\epsilon}} z_{2\boldsymbol{\epsilon}} & \text{if } \boldsymbol{\epsilon} = \boldsymbol{\eta} \\ 0 & \text{otherwise} \end{cases}$$

for all $\boldsymbol{\epsilon}, \boldsymbol{\eta} \in I$, where $0 \neq a_{\boldsymbol{\epsilon}} \in F$. Let

$$J = J_{S^{(m)}}(\{a_{\boldsymbol{\epsilon}}\}_{\boldsymbol{\epsilon}\in I}) := Z \oplus V$$

be the Jordan algebra over Z of f. Then

$$\{z_{\boldsymbol{\gamma}} \mid \boldsymbol{\gamma} \in \Gamma\} \cup \{z_{\boldsymbol{\gamma}} t_{\boldsymbol{\epsilon}} \mid \boldsymbol{\gamma} \in \Gamma,\ \boldsymbol{\epsilon} \in I\}$$

is an F-basis of J. For $\boldsymbol{\alpha} \in S^{(m)} \oplus \Lambda_{n-m}$, there exist unique $\boldsymbol{\alpha}' \in \Gamma$ and $\boldsymbol{\epsilon} \in I \cup \{\mathbf{0}\}$ such that $\boldsymbol{\alpha} = \boldsymbol{\alpha}' + \boldsymbol{\epsilon}$. Put $t_{\mathbf{0}} := 1$ and

$$t_{\boldsymbol{\alpha}} = \begin{cases} z_{\boldsymbol{\alpha}'} t_{\boldsymbol{\epsilon}} & \text{if } \boldsymbol{\alpha} \in S^{(m)} \oplus \Lambda_{n-m} \\ 0 & \text{otherwise.} \end{cases}$$

Then we get $J = \oplus_{\boldsymbol{\alpha}\in\Lambda} Ft_{\boldsymbol{\alpha}}$ as a graded F-vector space. By $(*)$, we have, for $\boldsymbol{\alpha} = \boldsymbol{\alpha}' + \boldsymbol{\epsilon},\ \boldsymbol{\beta} = \boldsymbol{\beta}' + \boldsymbol{\eta} \in S^{(m)} \oplus \Lambda_{n-m}$ where $\boldsymbol{\alpha}', \boldsymbol{\beta}' \in \Gamma$, $\boldsymbol{\epsilon}, \boldsymbol{\eta} \in I \cup \{0\}$,

$$(**) \qquad t_{\boldsymbol{\alpha}} t_{\boldsymbol{\beta}} = (z_{\boldsymbol{\alpha}'} t_{\boldsymbol{\epsilon}})(z_{\boldsymbol{\beta}'} t_{\boldsymbol{\eta}}) = \begin{cases} a_{\boldsymbol{\epsilon}} z_{\boldsymbol{\alpha}'+\boldsymbol{\beta}'} z_{2\boldsymbol{\epsilon}} = a_{\boldsymbol{\epsilon}} t_{\boldsymbol{\alpha}+\boldsymbol{\beta}} & \text{if } \boldsymbol{\epsilon} = \boldsymbol{\eta} \neq \mathbf{0} \\ z_{\boldsymbol{\alpha}'+\boldsymbol{\beta}'} t_{\boldsymbol{\epsilon}} = t_{\boldsymbol{\alpha}+\boldsymbol{\beta}} & \text{if } \boldsymbol{\eta} = \mathbf{0} \\ z_{\boldsymbol{\alpha}'+\boldsymbol{\beta}'} t_{\boldsymbol{\eta}} = t_{\boldsymbol{\alpha}+\boldsymbol{\beta}} & \text{if } \boldsymbol{\epsilon} = \mathbf{0} \\ 0 & \text{otherwise,} \end{cases}$$

and so we obtain $t_{\boldsymbol{\alpha}} t_{\boldsymbol{\beta}} \subset Ft_{\boldsymbol{\alpha}+\boldsymbol{\beta}}$ for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Lambda$. For $\boldsymbol{\alpha} = \boldsymbol{\alpha}' + \boldsymbol{\epsilon} \in S^{(m)} \oplus \Lambda_{n-m}$, since $t_{\boldsymbol{\alpha}}^2 = a_{\boldsymbol{\epsilon}} z_{2\boldsymbol{\alpha}'} z_{2\boldsymbol{\epsilon}}$ is invertible, $t_{\boldsymbol{\alpha}}$ is invertible. Since $\operatorname{supp} J = S^{(m)} \oplus \Lambda_{n-m}$, $\operatorname{supp} J$ generates Λ and hence $J = \oplus_{\boldsymbol{\alpha}\in\Lambda} Ft_{\boldsymbol{\alpha}}$ is a Jordan torus over F. One can check that the centre $Z(J)$ of J is Z and the central closure is a Jordan algebra of the extended bilinear form of f. Hence J is a Clifford torus.

We call $J = J_{S^{(m)}}(\{a_{\boldsymbol{\epsilon}}\}_{\boldsymbol{\epsilon}\in I})$ the *Clifford torus determined by* $S^{(m)}$ *of type* $\{a_{\boldsymbol{\epsilon}}\}_{\boldsymbol{\epsilon}\in I}$. The Λ-grading of J determined by a basis $\langle \boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n \rangle$ of Λ is called a *toral grading* or a $\langle \boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n \rangle$*-grading of* J. Also, when $a_{\boldsymbol{\epsilon}} = 1$ for all $\boldsymbol{\epsilon} \in I$, we call the $J = J_{S^{(m)}}$ the *standard Clifford torus determined by* $S^{(m)}$.

Remark 5.3. (1) One can easily check that

$$J_{S^{(m)}}(\{a_{\boldsymbol{\epsilon}}\}_{\boldsymbol{\epsilon}\in I}) \cong_\Lambda J' \otimes_F F[z_{m+1}^{\pm 1}, \dots, z_n^{\pm 1}]$$

where J' is the Clifford torus as constructed in 5.2 for $n = m$. When $m = n$, the standard Clifford torus $J_{S^{(n)}}$ appeared in [1] as the first example of an EALA of type A_1 graded by an arbitrary semilattice.

(2) If any element of F has a square root in F, e.g. F is an algebraically closed field, then one can make $a_{\boldsymbol{\epsilon}} = 1$ for all $\boldsymbol{\epsilon} \in I$ by switching $t_{\boldsymbol{\epsilon}}$ to $(\sqrt{a_{\boldsymbol{\epsilon}}})^{-1}t_{\boldsymbol{\epsilon}}$. Thus, for such a base field F, a Clifford torus $J = J_{S^{(m)}}(\{a_{\boldsymbol{\epsilon}}\}_{\boldsymbol{\epsilon}\in I})$ is always graded isomorphic to the standard Clifford torus $J_{S^{(m)}}$.

(3) A Clifford torus $J = J_{S^{(m)}}(\{a_{\boldsymbol{\epsilon}}\}_{\boldsymbol{\epsilon}\in I})$ is, by $(**)$, never of strong type, even if we take $S^{(m)} = \Lambda_m$.

We now start the classification. Let $J = \oplus_{\boldsymbol{\alpha}\in\Lambda} J_{\boldsymbol{\alpha}}$ be a Clifford torus over F, i.e., the central closure $\overline{J}$ is a Jordan algebra over $\overline{Z}$ of a symmetric bilinear form where $Z = Z(J)$ is the centre of J and $\overline{Z}$ is the field of fractions of Z. Thus $\overline{J}$ has degree ≤ 2 over $\overline{Z}$, i.e., there exist a $\overline{Z}$-linear form $\mathrm{tr} : \overline{J} \longrightarrow \overline{Z}$ and a $\overline{Z}$-quadratic map $n : \overline{J} \longrightarrow \overline{Z}$ with $n(1) = 1$ such that for all $x \in \overline{J}$,

$$x^2 - \mathrm{tr}(x)x + n(x)1 = 0.$$

If $\dim_{\overline{Z}} \overline{J} = 1$, then $J = Z$ since J embeds into $\overline{J}$. Hence J is a commutative associative torus, and so $J \cong_\Lambda F[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$.

Claim 1. *Let $J = \oplus_{\boldsymbol{\alpha}\in\Lambda} J_{\boldsymbol{\alpha}}$ be a Clifford torus such that $\dim_{\overline{Z}} \overline{J} \neq 1$. Let Γ be the central grading group of J. Then, for any $\boldsymbol{\alpha} \in \Lambda \setminus \Gamma$, we have $\mathrm{tr}(J_{\boldsymbol{\alpha}}) = \{0\}$, and there exists a basis $\{\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n\}$ of Λ such that*

$$\Gamma = 2\mathbb{Z}\boldsymbol{\sigma}_1 + \dots + 2\mathbb{Z}\boldsymbol{\sigma}_m + \mathbb{Z}\boldsymbol{\sigma}_{m+1} + \dots + \mathbb{Z}\boldsymbol{\sigma}_n \quad \textit{for some } 2 \leq m \leq n.$$

Proof. Recall that $\overline{J} = \oplus_{\overline{\boldsymbol{\alpha}}\in\Lambda/\Gamma} \overline{J}_{\overline{\boldsymbol{\alpha}}}$ is a Λ/Γ-graded algebra over $\overline{Z}$ (see 3.9(iii)). Note that $\dim_{\overline{Z}} \overline{J} \neq 1$ implies $\Lambda/\Gamma \neq \{\overline{\mathbf{0}}\}$, and so $\mathrm{supp}\, \overline{J} \neq \{\overline{\mathbf{0}}\}$. For any $\boldsymbol{\alpha} \in \mathrm{supp}\, J \setminus \Gamma$, let $0 \neq x \in J_{\boldsymbol{\alpha}}$. Then $x^2 + n(x)1 = \mathrm{tr}(x)x \in \overline{J}_{\overline{\boldsymbol{\alpha}}}$. If $\mathrm{tr}(x) \neq 0$, then $2\overline{\boldsymbol{\alpha}} = \overline{\boldsymbol{\alpha}}$ since $x^2 \neq 0$. Hence $\overline{\boldsymbol{\alpha}} = \overline{\mathbf{0}}$, which contradicts our choice of $\boldsymbol{\alpha}$. So we get $\mathrm{tr}(x) = 0$. Then $x^2 = -n(x)1 \in \overline{Z}1$. Since $0 \neq x^2 \in J_{2\boldsymbol{\alpha}} \cap \overline{Z}1 \subset \overline{J}_{2\overline{\boldsymbol{\alpha}}} \cap \overline{J}_{\overline{\mathbf{0}}}$, we get $2\overline{\boldsymbol{\alpha}} = \overline{\mathbf{0}}$.

Therefore, $2\Lambda \subset \Gamma$, and the exponent of Λ/Γ is 2, i.e., $\Lambda/\Gamma \cong \mathbb{Z}_2^m$ for some $1 \leq m \leq n$. By 3.6(iv), $m = 1$ cannot happen. Thus the statement is clear by the Fundamental Theorem of finitely generated abelian groups. $\square$

Let $W := \{x \in \overline{J} \mid \operatorname{tr}(x) = 0\}$. Then $\overline{J} = \overline{Z}1 \oplus W$ is a Jordan algebra over $\overline{Z}$ of the symmetric bilinear form

$$h := -\frac{1}{2}n(\cdot,\cdot)\mid_{W\times W}.$$

Recall that $J = \oplus_{\overline{\boldsymbol{\alpha}}\in\Lambda/\Gamma} J_{\overline{\boldsymbol{\alpha}}}$ is a Λ/Γ-graded algebra over Z in 3.9(ii). By Claim 1, we have $\operatorname{tr}(J_{\overline{\boldsymbol{\alpha}}}) = \{0\}$ for $\overline{\boldsymbol{\alpha}} \neq \overline{\mathbf{0}}$, and so

$$V := \oplus_{\overline{\boldsymbol{\alpha}}\neq\overline{\mathbf{0}}} J_{\overline{\boldsymbol{\alpha}}} \subset W$$

and $J = Z \oplus V$ as a direct sum of Z-modules. For all $x, y \in V$, we have $xy = h(x,y)1 \in J \cap \overline{Z}1 = J \cap Z(\overline{J}) = Z$. Therefore, $J = Z \oplus V$ is a Jordan algebra over Z of $f := h\mid_{V\times V}$. Let $S := \operatorname{supp} J$ and $\Lambda_m := \mathbb{Z}\boldsymbol{\sigma}_1 + \cdots + \mathbb{Z}\boldsymbol{\sigma}_m$ where $\{\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_n\}$ is a basis of Λ chosen in Claim 1.

Claim 2. *$S^{(m)} := S \cap \Lambda_m$ is a semilattice in Λ.*

Proof. Since S is a semilattice in Λ and Λ_m is a subgroup of Λ, we have $\mathbf{0}, 2\boldsymbol{\alpha} - \boldsymbol{\beta} \in S^{(m)}$ for all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in S^{(m)}$. We need to show that $S^{(m)}$ generates Λ_m. For any $\boldsymbol{\delta} = \delta_1\boldsymbol{\sigma}_1 + \cdots + \delta_n\boldsymbol{\sigma}_n \in S$, we have $\boldsymbol{\delta}' := \delta_{m+1}\boldsymbol{\sigma}_{m+1} + \cdots + \delta_n\boldsymbol{\sigma}_n \in \Gamma$. Let $0 \neq x \in J_{\boldsymbol{\delta}}$ and $0 \neq z \in J_{\boldsymbol{\delta}'}$. Then $0 \neq xz^{-1} \in J_{\boldsymbol{\delta}-\boldsymbol{\delta}'}$, and hence $\boldsymbol{\delta} - \boldsymbol{\delta}' = \delta_1\boldsymbol{\sigma}_1 + \cdots + \delta_m\boldsymbol{\sigma}_m \in S^{(m)}$. Since S generates Λ, we have for $\boldsymbol{\alpha} \in \Lambda_m \subset \Lambda$,

$$\boldsymbol{\alpha} = \sum_{\boldsymbol{\delta}\in U} l_{\boldsymbol{\delta}}\boldsymbol{\delta} = \sum_{\boldsymbol{\delta}\in U} l_{\boldsymbol{\delta}}(\boldsymbol{\delta} - \boldsymbol{\delta}')$$

where U is a finite subset of S and $l_{\boldsymbol{\delta}}$ is a positive integer. Therefore, $S^{(m)}$ generates Λ_m. $\square$

We fix a representative set I of $(S^{(m)}/2\Lambda_m) \setminus \{\overline{\mathbf{0}}\}$, as in 5.2. For $\boldsymbol{\epsilon} \in I$, let $0 \neq t_{\boldsymbol{\epsilon}} \in J_{\boldsymbol{\epsilon}}$. Then we get $V = \oplus_{\overline{\boldsymbol{\alpha}}\neq\overline{\mathbf{0}}} J_{\overline{\boldsymbol{\alpha}}} = \oplus_{\boldsymbol{\epsilon}\in I} Zt_{\boldsymbol{\epsilon}}$ as direct sums of Z-modules. Since $Z = \oplus_{\boldsymbol{\gamma}\in\Gamma} J_{\boldsymbol{\gamma}}$ is a commutative associative Γ-torus, Z can be identified with $F[z_1^{\pm1}, \ldots, z_n^{\pm1}]$ with the toral Γ-grading as in 5.2. Moreover, for $\boldsymbol{\epsilon} \neq \boldsymbol{\eta} \in I$, we have $\boldsymbol{\epsilon} + \boldsymbol{\eta} \notin \Gamma$, and so

$$t_{\boldsymbol{\epsilon}}t_{\boldsymbol{\eta}} = f(t_{\boldsymbol{\epsilon}}, t_{\boldsymbol{\eta}}) \in J_{\overline{\boldsymbol{\epsilon}+\boldsymbol{\eta}}} \cap J_{\overline{\mathbf{0}}} = (0).$$

Also, we have $0 \neq t_{\boldsymbol{\epsilon}}^2 = f(t_{\boldsymbol{\epsilon}}, t_{\boldsymbol{\epsilon}}) \in J_{2\boldsymbol{\epsilon}} = Fz_{2\boldsymbol{\epsilon}}$. Thus there exists $0 \neq a_{\boldsymbol{\epsilon}} \in F$ such that $f(t_{\boldsymbol{\epsilon}}, t_{\boldsymbol{\epsilon}}) = a_{\boldsymbol{\epsilon}}z_{2\boldsymbol{\epsilon}}$. Hence the bilinear form f coincides with f in 5.2. Consequently, we have shown the following:

Theorem 5.5. *Any $J_{S^{(m)}}(\{a_{\boldsymbol{\epsilon}}\}_{\boldsymbol{\epsilon}\in I})$ as defined in Example 5.2 is a Clifford n-torus and conversely any Clifford n-torus is graded isomorphic to $J_{S^{(m)}}(\{a_{\boldsymbol{\epsilon}}\}_{\boldsymbol{\epsilon}\in I})$, or to the algebra of Laurent polynomials in n variables, for some toral grading.*

Also, by Remark 5.3(2) we have:

Corollary 5.6. *Suppose that any element of the base field F has a square root. Then a Clifford torus is graded isomorphic to a standard Clifford n-torus $J_{S^{(m)}}$ as defined in 5.2, or to the algebra of Laurent polynomials in n variables, for some toral grading.* $\square$

§6 Albert type

We classify Jordan tori of Albert type, i.e., Jordan tori whose central closure is an Albert algebra. An Albert algebra is defined as either a *first* or a *second Tits construction*, which are both 27-dimensional central simple exceptional Jordan algebras of degree 3. We recall the first Tits construction but not the second one since second Tits constructions do not occur in the class of Jordan tori.

Let A be a central simple associative algebra over F of (generic) degree 3 with generic trace tr. For $a, b \in A$, let

$$\begin{aligned} a \cdot b &= \frac{1}{2}(ab+ba), \\ a \times b &= a\cdot b - \frac{1}{2}\operatorname{tr}(a)b - \frac{1}{2}\operatorname{tr}(b)a + \frac{1}{2}\big(\operatorname{tr}(a)\operatorname{tr}(b) - \operatorname{tr}(a\cdot b)\big)1, \\ \overline{a} &= a \times 1 = \frac{1}{2}\big(\operatorname{tr}(a)1 - a\big). \end{aligned}$$

Note that

$$\overline{1} = 1 \quad \text{and} \quad a(a\times a) = (a\times a)a = n(a)$$

where n is the generic norm on A.

Let $0 \neq \mu \in F$. A first Tits construction (A,μ) over F obtained from A and the structure constant μ is the direct sum $A\oplus A\oplus A$ as F-spaces with the following F-bilinear multiplication:

For $(a_0,a_1,a_2),(b_0,b_1,b_2) \in (A,\mu) = A\oplus A\oplus A$,

$$\begin{aligned} &(a_0,a_1,a_2)(b_0,b_1,b_2) = (a_0\cdot b_0 + \overline{a_1 b_2} + \overline{b_1 a_2}, \\ &\overline{a}_0 b_1 + \overline{b}_0 a_1 + \mu^{-1} a_2 \times b_2,\quad b_2\overline{a}_0 + a_2\overline{b}_0 + \mu a_1\times b_1) \end{aligned} \tag{6.1}$$

We will always identify A with $(A,0,0)$ as F-spaces.

Remark 6.2. Let $x := (0,1,0)$ and $y := (0,0,1)$. Since $a = \mathrm{tr}(a)1 + 2\overline{a} = \overline{\mathrm{tr}(a)1 + 2a}$, we have $A \cdot x = (0, A, 0)$ and $A \cdot x^2 = (0, 0, \mu A) = (0, 0, A)$. Thus (A, μ) is generated by A and x. Also, since $x = \mu y^2$, (A, μ) is generated by A and y.

The following lemma is well-known ([8] p.422, Exercise 1).

6.3. *Let (A, μ) be a first Tits construction. Let $a \in A$ be invertible and $x = (0, a, 0), y = (0, 0, a) \in (A, \mu)$. Then:*

(i) $0 \neq x^3 \in F1$*, and there exists an isomorphism Φ from (A, μ) onto (A, x^3) over $F = F1$ (identify) such that $\Phi\,|_A = id$ and $\Phi(x) = (0, 1, 0)$,*

(ii) $0 \neq y^{-3} \in F1$*, and there exists an isomorphism Ψ from (A, μ) onto (A, y^{-3}) over $F = F1$ such that $\Psi\,|_A = id$ and $\Psi(y) = (0, 0, 1)$.* $\square$

The theory of first Tits constructions over a ring does not seem to be much developed. As far as the author knows, the most general paper is [16]. For our purpose, we do not need this generality, but only a very special case. One might say that this is almost the classical case (i.e., Tits constructions over a field above).

Definition 6.4. We say that a prime Jordan or associative algebra P over F has *central degree* 3 if the central closure $\overline{P} = \overline{Z} \otimes_Z P$ is finite dimensional and has (generic) degree 3 over $\overline{Z}$.

Lemma 6.5. *Let A be a prime associative algebra over F of central degree 3, and $\mu \in Z$ a unit where $Z = Z(A)$ is the centre of A. Assume that $\mathrm{tr}(A) \subset Z$ where tr is the generic trace of the central closure $\overline{A}$ over $\overline{Z}$. Then, the subset $(A, \mu) := A \oplus A \oplus A$ of the first Tits construction $(\overline{A}, \mu) = \overline{A} \oplus \overline{A} \oplus \overline{A}$ is a Z-subalgebra such that $\overline{(A, \mu)} = (\overline{A}, \mu)$.*

Proof. By the multiplication rule of $(\overline{A}, \mu)$ (see 6.1), and our assumption $\mathrm{tr}(A) \subset Z$, it is clear that (A, μ) is a Z-subalgebra of $(\overline{A}, \mu)$. Let $Z(A, \mu)$ be the centre of (A, μ). Then we have $Z \subset Z(A, \mu) \subset Z(A) = Z$, and so $Z(A, \mu) = Z$. Therefore, the central closure $\overline{(A, \mu)}$ of (A, μ) is given as $\overline{Z} \otimes_Z (A, \mu)$, which is a $\overline{Z}$-subalgebra of $(\overline{A}, \mu)$. Thus we only need to show that $(\overline{A}, \mu) \subset \overline{(A, \mu)}$. But this is clear because we have, for $z_i \in Z$ and $a_i \in A$, $i = 0, 1, 2$,

$$\left(\frac{1}{z_0} \otimes a_0, \frac{1}{z_1} \otimes a_1, \frac{1}{z_2} \otimes a_2\right) = \frac{1}{z_0 z_1 z_2}(1 \otimes z_1 z_2 a_0, 1 \otimes z_0 z_2 a_1, 1 \otimes z_0 z_1 a_2)$$

Hence we get $(\overline{A}, \mu) = \overline{(A, \mu)}$. $\square$

We call this (A, μ) a *first Tits construction over* Z. This is a special type of the general first Tits construction studied in [16].

Remark 6.6. This (A, μ) is also generated as a Z-algebra by A and $x = (0, 1, 0)$ or by A and $y = (0, 0, 1)$ as in the classical case (see 6.2). Thus (A, μ) is characterized as the Z-subalgebra of $(\overline{A}, \mu)$ generated by A and x or by A and y.

Before giving examples of Jordan tori of central degree 3, we show general properties of them.

Proposition 6.7. *Let* $T = \oplus_{\boldsymbol{\alpha} \in \Lambda} T_{\boldsymbol{\alpha}}$ *be a division* Λ*-graded Jordan or associative algebra over* F *of central degree* 3*. Let* tr *be the generic trace of the central closure* $\overline{T}$*. Then there exists a basis* $\{\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n\}$ *of* Λ *such that the central grading group of* T *is given as*

$$\Gamma = 3\mathbb{Z}\boldsymbol{\sigma}_1 + \cdots + 3\mathbb{Z}\boldsymbol{\sigma}_m + \mathbb{Z}\boldsymbol{\sigma}_{m+1} + \cdots + \mathbb{Z}\boldsymbol{\sigma}_n \quad \text{for some } 2 \leq m \leq n,$$

and $\operatorname{supp} T = \Lambda$*. Moreover, for any* $\boldsymbol{\alpha} \in \Lambda \setminus \Gamma$*, we have* $\operatorname{tr}(T_{\boldsymbol{\alpha}}) = \{0\}$*.*

Proof. If $\Gamma = \Lambda$, then $\dim_{\overline{Z}} \overline{T} = 1$, and hence T does not have central degree 3. Therefore, $\Gamma \neq \Lambda$ and $\operatorname{supp} T/\Gamma \neq \{0\}$. Let $\overline{\mathbf{0}} \neq \overline{\boldsymbol{\beta}} \in \operatorname{supp} T/\Gamma$ and $0 \neq x \in \overline{T}_{\overline{\boldsymbol{\beta}}}$. Since $\overline{T} = \oplus_{\overline{\boldsymbol{\alpha}} \in \Lambda/\Gamma} \overline{T}_{\overline{\boldsymbol{\alpha}}}$ (see 3.9(iii)), has generic degree 3, we have

$$x^3 + z_1 x^2 + z_2 x + z_3 1 = 0 \quad \text{for some } z_1, z_2, z_3 \in \overline{Z} \text{ and } z_1 = -\operatorname{tr}(x).$$

If $\overline{2\boldsymbol{\beta}} = \overline{\mathbf{0}}$, then $\overline{3\boldsymbol{\beta}} = \overline{\boldsymbol{\beta}}$ and therefore,

$$x^3 + z_2 x = -z_1 x^2 - z_3 1 \in \overline{T}_{\overline{\boldsymbol{\beta}}} \cap \overline{T}_{\overline{\mathbf{0}}} = (0).$$

Hence we get $x^3 + z_2 x = x(x^2 + z_2 1) = 0$. Since $\overline{T}$ is a Jordan or an associative domain, the subalgebra $\overline{Z}[x]$ of $\overline{T}$ generated by x is a commutative associative algebra domain over $\overline{Z}$. So $x^2 + z_2 1 = 0$ since $x \neq 0$. Since $x \notin \overline{T}_{\overline{\mathbf{0}}}$, the polynomial $f(\lambda) = \lambda^2 + z_2$ is the minimal polynomial of x over $\overline{Z}$. If $f(\lambda)$ is reducible over $\overline{Z}$, say $f(\lambda) = (\lambda - a)(\lambda - b)$, $a, b \in \overline{Z}$, then $(x - a1)(x - b1) = 0$ in $\overline{Z}[x]$. Hence, $x = a1$ or $x = b1$, and so $x \in \overline{Z}1 = \overline{T}_{\overline{\mathbf{0}}}$, i.e., $\overline{\boldsymbol{\beta}} = \overline{\mathbf{0}}$, which is a contradiction. Therefore, $f(\lambda)$ is irreducible over $\overline{Z}$. Note that the minimal polynomial and the generic minimal polynomial of an element have the same irreducible factors (see [8] p.224). Since $f(\lambda)$ is the irreducible minimal polynomial of x, the generic minimal polynomial of x has to be a power of

$f(\lambda)$. However, this is impossible since the degree of the generic minimal polynomial of x is 3. Therefore, $\overline{2\boldsymbol{\beta}} \neq \overline{\mathbf{0}}$. This implies that $\overline{3\boldsymbol{\beta}} \neq \overline{\boldsymbol{\beta}}$. Since $\overline{\boldsymbol{\beta}} \neq \overline{\mathbf{0}}$, we have $\overline{3\boldsymbol{\beta}} \neq \overline{2\boldsymbol{\beta}}$. Hence $\{\overline{3\boldsymbol{\beta}}, \overline{\mathbf{0}}\} \cap \{\overline{2\boldsymbol{\beta}}, \overline{\boldsymbol{\beta}}\} = \emptyset$. So $(\overline{T}_{\overline{3\boldsymbol{\beta}}} + \overline{T}_{\overline{\mathbf{0}}}) \cap (\overline{T}_{\overline{2\boldsymbol{\beta}}} \oplus \overline{T}_{\overline{\boldsymbol{\beta}}}) = (0)$. Since

$$x^3 + z_3 1 = -z_1 x^2 - z_2 x \in (\overline{T}_{\overline{3\boldsymbol{\beta}}} + \overline{T}_{\overline{\mathbf{0}}}) \cap (\overline{T}_{\overline{2\boldsymbol{\beta}}} \oplus \overline{T}_{\overline{\boldsymbol{\beta}}}),$$

we get two equalities $x^3 + z_3 1 = 0$ and $-z_1 x^2 - z_2 x = 0$.

By the first identity, we have $0 \neq x^3 = -z_3 1 \in \overline{T}_{\overline{3\boldsymbol{\beta}}} \cap \overline{T}_{\overline{\mathbf{0}}}$, and hence $\overline{3\boldsymbol{\beta}} = \overline{\mathbf{0}}$. Thus $3\Lambda \subset \Gamma$, and so the exponent of Λ/Γ is 3. Hence, by 3.9(iv), $\Lambda/\Gamma \cong \mathbb{Z}_3^m$ for some $2 \leq m \leq n$, and so the first statement follows from the Fundamental Theorem of finitely generated abelian groups. Also, we have $3\Lambda \subset \operatorname{supp} T$. Since $\operatorname{supp} T$ is a semilattice, $\Lambda = 3\Lambda - 2\Lambda \subset \operatorname{supp} T$, and so $\operatorname{supp} T = \Lambda$.

By the second identity and by the same reason above, we have $-z_1 x - z_2 1 = 0$. Then $-z_1 x = z_2 1 \in \overline{T}_{\overline{\boldsymbol{\beta}}} \cap \overline{T}_{\overline{\mathbf{0}}} = (0)$. Hence $z_1 = 0$, i.e., $\operatorname{tr}(x) = 0$. Therefore, for any $\boldsymbol{\alpha} \in \Lambda \setminus \Gamma$, we have $\operatorname{tr}(T_{\boldsymbol{\alpha}}) = \{0\}$. $\square$

We give examples of Jordan tori of central degree 3.

Example 6.8. (1) Assume that F contains a primitive 3rd root of unity ω. Let $\boldsymbol{\omega}$ be an $n \times n$ ($n \geq 2$) quantum matrix

$$\boldsymbol{\omega} = \boldsymbol{\omega}_n = \begin{pmatrix} 1 & \omega & 1 & \cdots & 1 \\ \omega^{-1} & 1 & 1 & & \vdots \\ 1 & 1 & 1 & & \vdots \\ \vdots & & & \ddots & 1 \\ 1 & \cdots & \cdots & 1 & 1 \end{pmatrix}$$

where the $(1,2)$-entry is ω, the $(2,1)$-entry is ω^{-1} and the other entries are all 1. Let $F_{\boldsymbol{\omega}} = F_{\boldsymbol{\omega}}[u_1^{\pm 1}, \ldots, u_n^{\pm 1}]$ be the quantum torus determined by $\boldsymbol{\omega}$ and $Z = Z(F_{\boldsymbol{\omega}})$ the centre of $F_{\boldsymbol{\omega}}$. One finds that $Z = F[u_1^{\pm 3}, u_2^{\pm 3}, u_3^{\pm 1}, \ldots, u_n^{\pm 1}]$, the algebra of Laurent polynomials in the variables $u_1^3, u_2^3, u_3, \ldots, u_n$. So for a $\langle \boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_n \rangle$-grading of $F_{\boldsymbol{\omega}}$, the central grading group of $F_{\boldsymbol{\omega}}$ is $3\mathbb{Z}\boldsymbol{\sigma}_1 + 3\mathbb{Z}\boldsymbol{\sigma}_2 + \mathbb{Z}\boldsymbol{\sigma}_3 + \cdots + \mathbb{Z}\boldsymbol{\sigma}_n$, and the central closure $\overline{F}_{\boldsymbol{\omega}}$ is a $(\mathbb{Z}_3 \times \mathbb{Z}_3)$-torus over $\overline{Z}$. So the dimension of $\overline{F}_{\boldsymbol{\omega}}$ over $\overline{Z}$ is 9. Since $\overline{F}_{\boldsymbol{\omega}}$ is a domain, $\overline{F}_{\boldsymbol{\omega}}$ is a division algebra, by Wedderburn's Structure Theorem, and $\overline{F}_{\boldsymbol{\omega}}$ has degree 3 over $\overline{Z}$. Thus $F_{\boldsymbol{\omega}}$ has central degree 3. We claim that $F_{\boldsymbol{\omega}}^+$ is a special Jordan torus of central degree 3. In fact, we have, by 2.5, $Z(F_{\boldsymbol{\omega}}^+) = Z(F_{\boldsymbol{\omega}})$. So the central closure $\overline{F_{\boldsymbol{\omega}}^+}$ is a 9-dimensional central special Jordan division algebra over $\overline{Z(F_{\boldsymbol{\omega}}^+)} = \overline{Z(F_{\boldsymbol{\omega}})}$. Hence, by 2.11(b), it has degree 3.

(2) Assume that $n \geq 3$. Let $F_{\boldsymbol{\omega}} = F_{\boldsymbol{\omega}}[u_1^{\pm 1}, \dots, u_n^{\pm 1}]$ as defined in (1), $Z(F_{\boldsymbol{\omega}})$ its centre and tr the generic trace of the central closure $\overline{F}_{\boldsymbol{\omega}}$. Then, by 6.7, we have $\mathrm{tr}(u_1^i u_2^j) = 0$ if $i \not\equiv 0$ or $j \not\equiv 0 \pmod 3$, and so

$$\mathrm{tr}(F_{\boldsymbol{\omega}}) \subset Z(F_{\boldsymbol{\omega}}) = F[u_1^{\pm 3}, u_2^{\pm 3}, u_3^{\pm 1}, \dots, u_n^{\pm 1}].$$

Thus, by 6.5, we have the first Tits construction $\mathbb{A}_t = (F_{\boldsymbol{\omega}}, u_3)$ over Z. Namely,

$$\mathbb{A}_t = (F_{\boldsymbol{\omega}}, u_3) = F_{\boldsymbol{\omega}} \oplus F_{\boldsymbol{\omega}} \oplus F_{\boldsymbol{\omega}}.$$

Let $\langle \boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n \rangle$ be a basis of Λ and

$$\Delta := \mathbb{Z}\boldsymbol{\sigma}_1 + \mathbb{Z}\boldsymbol{\sigma}_2 + 3\mathbb{Z}\boldsymbol{\sigma}_3 + \mathbb{Z}\boldsymbol{\sigma}_4 + \cdots + \mathbb{Z}\boldsymbol{\sigma}_n.$$

We give $F_{\boldsymbol{\omega}}$ a toral Δ-grading, i.e.,

$$F_{\boldsymbol{\omega}} = \oplus_{\boldsymbol{\delta} \in \Delta} \, F u_{\boldsymbol{\delta}} \quad \text{where} \quad u_{\boldsymbol{\delta}} = u_1^{\delta_1} u_2^{\delta_2} u_3^{\delta_3} \cdots u_n^{\delta_n}$$

for $\boldsymbol{\delta} = \delta_1 \boldsymbol{\sigma}_1 + \delta_2 \boldsymbol{\sigma}_2 + 3\delta_3 \boldsymbol{\sigma}_3 + \delta_4 \boldsymbol{\sigma}_4 + \cdots + \delta_n \boldsymbol{\sigma}_n$.

For $\boldsymbol{\alpha} = \alpha_1 \boldsymbol{\sigma}_1 + \cdots + \alpha_n \boldsymbol{\sigma}_n \in \Lambda$, we put

$$t_{\boldsymbol{\alpha}} := \begin{cases} (u_{\boldsymbol{\alpha}}, 0, 0) & \text{if } \alpha_3 \equiv 0 \pmod 3 \\ (0, u_{\boldsymbol{\alpha} - \boldsymbol{\sigma}_3}, 0) & \text{if } \alpha_3 \equiv 1 \pmod 3 \\ (0, 0, u_{\boldsymbol{\alpha} + \boldsymbol{\sigma}_3}) & \text{if } \alpha_3 \equiv 2 \pmod 3). \end{cases}$$

Then we obtain $t_{\boldsymbol{\alpha}} \neq 0$ for all $\boldsymbol{\alpha} \in \Lambda$ and $\mathbb{A}_t = \oplus_{\boldsymbol{\alpha} \in \Lambda} F t_{\boldsymbol{\alpha}}$. Thus, $\mathbb{A}_t$ is a Λ-graded vector space over F whose homogeneous spaces are all 1-dimensional over F. We note that $t_{\boldsymbol{\sigma}_3} = (0, 1, 0)$, $t_{2\boldsymbol{\sigma}_3} = t_{\boldsymbol{\sigma}_3}^2 = (0, 0, u_3)$ and $t_{-\boldsymbol{\sigma}_3} = t_{\boldsymbol{\sigma}_3}^{-1} = (0, 0, 1)$. By the multiplication rule of Tits first constructions (see 6.1), one can check that $\mathbb{A}_t$ is a Λ-graded algebra and the structure constants relative to the basis $\{t_{\boldsymbol{\alpha}}\}_{\boldsymbol{\alpha} \in \Lambda}$ are

$$\{1, \omega, \omega^2, -\frac{1}{2}, -\frac{\omega}{2}, -\frac{\omega^2}{2}\}.$$

Hence $\mathbb{A}_t$ is a Jordan torus over F of strong type, which is called the *Albert torus over F*. We call the grading of $\mathbb{A}_t$ above a *toral grading* or a $\langle \boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n \rangle$*-grading*. By 6.5, the central closure $\overline{\mathbb{A}}_t$ of $\mathbb{A}_t$ is an Albert algebra $(\overline{F}_{\boldsymbol{\omega}}, u_3)$ over $\overline{Z}$, and so the Albert torus is in fact a Jordan torus of Albert type. By 2.10, the central closure of a Jordan torus of Albert type is a division algebra since this central closure is a 27-dimensional Jordan domain. In particular, the central closure $\overline{\mathbb{A}_t}$ is a division algebra.

(3) Suppose that $\sqrt{-3} \notin F$. Let $E = F(\sqrt{-3})$ and consider the Jordan torus $H(E_{\boldsymbol{\omega}}, \sigma)$ defined in 4.3(3). Since $E \otimes_F H(E_{\boldsymbol{\omega}}, \sigma) \cong_\Lambda E_{\boldsymbol{\omega}}^+$, the central grading group of $H(E_{\boldsymbol{\omega}}, \sigma)$ coincides with the one of $E_{\boldsymbol{\omega}}$ (see 3.8), which is $3\mathbb{Z}\boldsymbol{\sigma}_1 + 3\mathbb{Z}\boldsymbol{\sigma}_2 + \mathbb{Z}\boldsymbol{\sigma}_3 + \cdots + \mathbb{Z}\boldsymbol{\sigma}_n$ by (1). Therefore, the central closure $\overline{H(E_{\boldsymbol{\omega}}, \sigma)}$ is a 9-dimensional central special Jordan algebra, which by 2.10 is a division algebra. Then, by 2.11(b), $H(E_{\boldsymbol{\omega}}, \sigma)$ has central degree 3.

We first classify associative tori of central degree 3. Let $T = \oplus_{\boldsymbol{\alpha} \in \Lambda} T_{\boldsymbol{\alpha}}$ be an associative torus over F of central degree 3 and $\overline{T}$ the central closure over $\overline{Z}$. By 6.7, we have $\dim_{\overline{Z}} \overline{T} = 3^m$ for some $2 \le m \le n$. Since $\overline{T}$ is a finite dimensional associative domain, $\overline{T}$ is a division algebra by Wedderburn's Structure Theorem. Hence $m = 2$, i.e., $\dim_{\overline{Z}} \overline{T} = 9$. Thus there exists a basis $\{\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_n\}$ of Λ such that the central grading group Γ of T is $3\mathbb{Z}\boldsymbol{\sigma}_1 + 3\mathbb{Z}\boldsymbol{\sigma}_2 + \mathbb{Z}\boldsymbol{\sigma}_3 + \cdots + \mathbb{Z}\boldsymbol{\sigma}_n$. Also, it is clear that an associative torus whose central grading group is Γ has central degree 3.

Now, we classify associative tori whose central grading group is Γ. Let $0 \ne t_i \in \mathcal{T}_{\boldsymbol{\sigma}_i}$ for $1 \le i \le n$. Then since $t_i t_1 = t_1 t_i$, $t_i t_2 = t_2 t_i$ and $t_j t_i = t_i t_j$ for all $3 \le i \le n$ and $1 \le j \le n$, we can identify such a T with the quantum torus $F_{\boldsymbol{q}} = F_{\boldsymbol{q}}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ determined by $\boldsymbol{q} = \boldsymbol{q}(q)$ where the $(1,2)$-entry of $\boldsymbol{q}$ is some $q \in F^\times$, the $(2,1)$-entry is q^{-1} and the other entries are all 1. Moreover, since $t_2^3 \in Z$, we have $t_1 t_2^3 = t_2^3 t_1 = q^3 t_1 t_2^3$, and so $q^3 = 1$. If $q = 1$, then $\boldsymbol{q} = \boldsymbol{q}(1) = \mathbf{1}$, but the algebra of Laurent polynomials $F_{\mathbf{1}}$ cannot have central degree 3 since $Z = Z(F_{\mathbf{1}}) = F_{\mathbf{1}}$. Hence $q \ne 1$, and F has to contain a primitive 3rd root of unity, say ω. Since q can be either ω or ω^{-1}, let $\boldsymbol{q} := \boldsymbol{q}(\omega)$ and $\boldsymbol{q}' := \boldsymbol{q}(\omega^{-1})$. One can easily see that $F_{\boldsymbol{q}} \cong F_{\boldsymbol{q}'}$ via $t_1 \mapsto t_2$, $t_2 \mapsto t_1$ and $t_i \mapsto t_i$ for $i = 3, \ldots, n$.

Remark 6.11. Note that $F_{\boldsymbol{q}'}$ can be identified with the opposite algebra $F_{\boldsymbol{q}}^{op}$ of $F_{\boldsymbol{q}}$. Then $F_{\boldsymbol{q}}$ and $F_{\boldsymbol{q}'}$ are both algebras over their common centre

$$Z = F[t_1^{\pm 3}, t_2^{\pm 3}, t_3^{\pm 1} \ldots, t_n^{\pm 1}].$$

We showed that $F_{\boldsymbol{q}} \cong F_{\boldsymbol{q}'}$ over F, but we note that $F_{\boldsymbol{q}} \ncong F_{\boldsymbol{q}'}$ over Z. In general, if A is an associative domain of central degree 3, then we always have $A \ncong A^{op}$ over Z. For, if $A \cong A^{op}$ over Z, then $\overline{A} \cong \overline{A}^{op}$ over $\overline{Z}$, which cannot happen since $\overline{A}$ is a central associative division algebra of degree 3. (See e.g. [19]; if $\overline{A} \cong \overline{A}^{op}$, then

$[\overline{A}]^2 = 1$ in the Brauer group. But the order of $\overline{A}$ in the Brauer group has to divide the degree.)

We summarize the above results as a proposition.

Proposition 6.12. (1) *For an associative torus T over F we have: T has central degree 3 $\Longleftrightarrow$ the central grading group of T is $3\mathbb{Z}\boldsymbol{\sigma}_1 + 3\mathbb{Z}\boldsymbol{\sigma}_2 + \mathbb{Z}\boldsymbol{\sigma}_3 + \cdots + \mathbb{Z}\boldsymbol{\sigma}_n$ for some basis $\{\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n\}$ of Λ.*

(2) *If $\omega \in F$, then an associative torus over F of central degree 3 exists, and any such torus is isomorphic to the quantum torus $F_{\boldsymbol{\omega}}$. If $\omega \notin F$, e.g.* ch. $F = 3$, *no such torus exists.* $\square$

Unlike the other types, we will show that the Albert torus $\mathbb{A}_t$ is the only Jordan torus of Albert type. We begin with the following:

Proposition 6.13. *Let J be a special Jordan torus over F of central degree 3. Then* ch. $F \neq 3$, *and*

$$J \cong_\Lambda \begin{cases} F_{\boldsymbol{\omega}}^+ & \text{if } \omega \in F \\ H(E_{\boldsymbol{\omega}}, \sigma) & \text{otherwise} \end{cases}$$

where $E = F(\sqrt{-3})$. Conversely, the algebras $F_{\boldsymbol{\omega}}^+$ and $H(E_{\boldsymbol{\omega}}, \sigma)$ are special Jordan tori of central degree 3.

Remark. If ch. $F \neq 3$, there exists a primitive 3rd root of unity ω in some extension field of F. Note that $F(\omega) = F(\sqrt{-3})$.

Proof. Only the 1st part remains to be proven. We know that a special Jordan torus J is either a Hermitian torus or a Clifford torus. Since a Clifford torus does not have central degree 3, special Jordan tori of central degree 3 have to be Hermitian tori, i.e., $J \cong_\Lambda F_{\boldsymbol{q}}^+$, $H(F_{\boldsymbol{\varepsilon}}, *)$ or $H(E_{\boldsymbol{\xi}}, \sigma)$ (see 4.3 and 4.11). Let $Z = Z(J)$ be the centre of J. By 6.7, there exists a basis $\{\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n\}$ of Λ such that the central grading group Γ of J has the form

$$\Gamma = 3\mathbb{Z}\boldsymbol{\sigma}_1 + \cdots + 3\mathbb{Z}\boldsymbol{\sigma}_m + \mathbb{Z}\boldsymbol{\sigma}_{m+1} + \cdots + \mathbb{Z}\boldsymbol{\sigma}_n$$

for some $2 \leq m \leq n$ and $\operatorname{supp} J = \Lambda$. In particular, there exists a homogeneous element $t \in J$ such that $t^2 \notin Z$ (for example, take t to be a nonzero element of degree $\boldsymbol{\sigma}_1$). If $\varphi : J \xrightarrow{\sim} H(F_{\boldsymbol{\varepsilon}}, *)$ is a graded isomorphism, then by 3.6(ii), $\varphi(t)$ is homogeneous in $H(F_{\boldsymbol{\varepsilon}}, *)$, and $\varphi(t)^2 \notin \varphi(Z) = Z\big(H(F_{\boldsymbol{\varepsilon}}, *)\big)$. However, for any homogeneous element $x \in H(F_{\boldsymbol{\varepsilon}}, *)$, $x^2 \in Z\big(H(F_{\boldsymbol{\varepsilon}}, *)\big)$ since the entries of $\boldsymbol{\varepsilon}$ are ± 1. Thus we get a contradiction. Hence the only possible candidates of central degree 3 are $F_{\boldsymbol{q}}^+$ and $H(E_{\boldsymbol{\xi}}, \sigma)$. Since the central closure $\overline{J}$ is a finite dimensional central special Jordan division algebra over $\overline{Z}$ of degree 3, we have, by 2.11(b), $\dim_{\overline{Z}} \overline{J} = 9$. So m has to be 2, by 3.9(iii).

If $J \cong_\Lambda F_{\boldsymbol{q}}^+$, then by 2.5, $Z = Z(F_{\boldsymbol{q}}^+) = Z(F_{\boldsymbol{q}})$, and hence the central grading group of $F_{\boldsymbol{q}}$ is Γ. Thus, by 6.12, we have $\omega \in F$, and so $\mathrm{ch.}\, F \neq 3$, and $\boldsymbol{q} = \boldsymbol{\omega}$, i.e., $J \cong_\Lambda F_{\boldsymbol{\omega}}^+$.

Suppose that $J \cong_\Lambda H(E_{\boldsymbol{\xi}}, \sigma)$. We identify them. So we have $J_E = E \otimes_F J = E_{\boldsymbol{\xi}}^+$. Hence, by 2.5 and 3.8, the grading group of $Z(E_{\boldsymbol{\xi}}^+) = Z(E_{\boldsymbol{\xi}})$ is Γ. Then, by 6.12, we get $\mathrm{ch.}\, E \neq 3$, $\omega \in E$ and $\boldsymbol{\xi} = \boldsymbol{\omega}$, i.e., $J = H(E_{\boldsymbol{\omega}}, \sigma)$. Since $\omega\sigma(\omega) = 1$, we have $\sigma(\omega) \neq \omega$, and so $\omega \notin F$. Since $[E : F] = 2$, we obtain $E = F(\omega) = F(\sqrt{-3})$. $\square$

By Zelmanov's Prime Structure Theorem, if $\mathcal{J}$ is a strongly prime exceptional Jordan algebra $\mathcal{J}$ over F, then $\mathcal{J}$ has central degree 3 and the central closure $\overline{\mathcal{J}}$ is an Albert algebra. Let Tr be the generic trace of $\overline{\mathcal{J}}$ over $\overline{Z}$. For a subalgebra $\mathcal{U}$ of $\mathcal{J}$, let

$$\mathcal{U}^\perp := \{x \in \mathcal{J} \mid \mathrm{Tr}(\mathcal{U}x) = 0\} \subset \mathcal{J} \subset \overline{\mathcal{J}}.$$

Note that the central closure of an exceptional Jordan domain is an Albert division algebra. The following lemma for an exceptional Jordan domain serves as preparation for the classification of Jordan tori of Albert type. We will show that such a torus satisfies all the assumptions of the lemma.

Lemma 6.14. *Let $\mathcal{J}$ be an exceptional Jordan domain over F, $Z = Z(\mathcal{J})$ the centre of $\mathcal{J}$ and* Tr *the generic trace of $\overline{\mathcal{J}}$. Let $\mathcal{U}$ be a subdomain of $\mathcal{J}$ and $Z(\mathcal{U})$ the centre of $\mathcal{U}$. We assume the following conditions:*

(i) $Z = Z(\mathcal{U})$ *and $\mathcal{U}$ has central degree* 3*,*

(ii) $\mathcal{U} = \mathcal{A}^+$ *for some associative algebra $\mathcal{A}$ over F,*

(iii) $\mathrm{Tr}(\mathcal{U}) \subset Z$*,*

(iv) *there exists an element $x \in \mathcal{U}^\perp$ such that $x^2 \in \mathcal{U}^\perp$ and $z := x^3 \in Z$ is invertible.*

Then, $\mathcal{J}$ contains a subalgebra $\mathcal{J}'$ so that there exists

$$\text{a } Z\text{-isomorphism } \varphi : (\mathcal{A}, z) \xrightarrow{\sim} \mathcal{J}' \text{ with } \varphi|_{\mathcal{A}} = \mathrm{id} \text{ and } \varphi\big((0,1,0)\big) = x \text{ or,}$$
$$\text{a } Z\text{-isomorphism } \psi : (\mathcal{A}, z^{-1}) \xrightarrow{\sim} \mathcal{J}' \text{ with } \psi|_{\mathcal{A}} = \mathrm{id} \text{ and } \psi\big((0,0,1)\big) = x,$$

where $(\mathcal{A}, z)$ and $(\mathcal{A}, z^{-1})$ are defined in 6.5.

Moreover, assume that

(v) *there exists an F-isomorphism f from $\mathcal{A}$ onto the opposite algebra $\mathcal{A}^{op}$ such that $f \circ \mathrm{Tr} = \mathrm{Tr} \circ f$ and $f(z) = z$.*

Then there exists an F-isomorphism $\tilde{f} : (\mathcal{A}, z) \xrightarrow{\sim} (\mathcal{A}, z^{-1})$ with $\tilde{f} \mid_{\mathcal{A}} = f$ and $\tilde{f}\big((0,1,0)\big) = (0,0,1)$. In particular, $\mathcal{J}$ always contains a subalgebra F-isomorphic to $(\mathcal{A}, z)$.

Proof. As mentioned above, the central closure $\overline{\mathcal{J}}$ is an Albert division algebra over $\overline{Z}$. By (i), $\overline{\mathcal{U}} = \overline{Z(\mathcal{U})} \otimes_{Z(\mathcal{U})} \mathcal{U} = \overline{Z} \otimes_Z \mathcal{U} \subset \overline{\mathcal{J}}$ is a central division subalgebra over $\overline{Z}$ of degree 3, and $\mathrm{tr} := \mathrm{Tr} \mid_{\overline{\mathcal{U}}}$ is the generic trace of $\overline{\mathcal{U}}$. By (ii) and 2.5, we have

$$\overline{\mathcal{U}} = \overline{Z(\mathcal{A}^+)} \otimes_{Z(\mathcal{A}^+)} \mathcal{A}^+ = \overline{Z(\mathcal{A})} \otimes_{Z(\mathcal{A})} \mathcal{A}^+ = \big(\overline{Z(\mathcal{A})} \otimes_{Z(\mathcal{A})} \mathcal{A}\big)^+ = (\overline{\mathcal{A}})^+.$$

Hence $B := \overline{\mathcal{A}}$ is a central associative division algebra over $\overline{Z}$ of degree 3, and so $\mathcal{A}$ has central degree 3 with $\mathrm{tr}(\mathcal{A}) = \mathrm{tr}(\mathcal{U}) = \mathrm{Tr}(\mathcal{U}) \subset Z$ by (iii). Note that the generic trace of $\mathcal{A}$ coincides with the generic trace tr of $\mathcal{U} = \mathcal{A}^+$ (see [8] p.230).

Now, since $\overline{\mathcal{J}}$ contains $\overline{\mathcal{U}} = B^+$ for the central simple associative algebra B over $\overline{Z}$ of degree 3, we have $\overline{\mathcal{J}} \cong (B, \mu)$ over $\overline{Z}$ for some $0 \neq \mu \in \overline{Z}$ (see [8] p.420). Note that this isomorphism is the identity map on B. So we identify $\overline{\mathcal{J}}$ with (B, μ). Since $\overline{\mathcal{U}}^{\perp} = B^{\perp} = (0, B, B)$ (see e.g. [18] p.349), we have, by (iv), $x = (0, u, v)$ for some $u, v \in B$. By the multiplication rule 6.1 of (B, μ), we have $x^2 = (2\overline{uv}, u', v')$ for some $u', v' \in B$. Since $x^2 \in \mathcal{U}^{\perp}$, we have $2\overline{uv} = 0$, and so $\mathrm{tr}(uv)1 = uv$. Hence $0 = 2\overline{uv} = 2\overline{\mathrm{tr}(uv)1} = 2\,\mathrm{tr}(uv)1$, and so $uv = 0$. Since B is an associative division algebra, we have $u = 0$ or $v = 0$. If $u = v = 0$, then $x = 0$ which contradicts the invertibility of x^3. Thus we obtain $x = (0, u, 0)$ or $x = (0, 0, v)$ for some nonzero $u, v \in B$. Then, by 6.3, there exists a $\overline{Z}$-isomorphism Φ from (B, μ) onto (B, z) or a $\overline{Z}$-isomorphism Ψ from (B, μ) onto (B, z^{-1}) such that $\Phi \mid_{\mathcal{A}} = \mathrm{id}$ or $\Psi \mid_{\mathcal{A}} = \mathrm{id}$ and $\Phi(x) = (0, 1, 0)$ or $\Psi(x) = (0, 0, 1)$. So, $\Phi(\mathcal{J})$ contains $\mathcal{A}$ and $(0, 1, 0)$ in (B, z), or $\Psi(\mathcal{J})$ contains $\mathcal{A}$ and $(0, 0, 1)$ in (B, z^{-1}). Hence $\Phi(\mathcal{J})$ contains $(\mathcal{A}, z)$ or $\Psi(\mathcal{J})$ contains $(\mathcal{A}, z^{-1})$ (see 6.6). Let $\mathcal{J}' = \Phi^{-1}\big((\mathcal{A}, z)\big)$ or $\mathcal{J}' = \Psi^{-1}\big((\mathcal{A}, z^{-1})\big)$. Then $\varphi := \Phi^{-1} \mid_{(\mathcal{A}, z)}$ and $\psi := \Psi^{-1} \mid_{(\mathcal{A}, z^{-1})}$ are the required Z-isomorphisms, and so we have shown the first statement.

For the second statement, we use the well-known fact that there exists an isomorphism $g : (B, z) \xrightarrow{\sim} (B^{op}, z^{-1})$ over $\overline{Z}$ defined by $g\big((a_0, a_1, a_2)\big) = (a_0, a_2, a_1)$ for $a_0, a_1, a_2 \in \overline{\mathcal{A}}$ (see [8] p.422, Exercise 2). Thus

$$h := g \mid_{(\mathcal{A}, z)} : (\mathcal{A}, z) \xrightarrow{\sim} (\mathcal{A}^{op}, z^{-1})$$

is a Z-isomorphism. We note that $\mathcal{A}^{op}$ has central degree 3, and has the same centre as $\mathcal{A}$. Also, the generic trace of $\mathcal{A}^{op}$ coincides with the generic trace tr of $\mathcal{A}$. The F-isomorphism $f : \mathcal{A} \xrightarrow{\sim} \mathcal{A}^{op}$ in our assumption (v) satisfies $f(z) = z$ and $f \circ \mathrm{tr} = f \circ \mathrm{Tr} = \mathrm{Tr} \circ f = \mathrm{tr} \circ f$. So one can check that the map $\bar{f} : (\mathcal{A}, z^{-1}) \longrightarrow (\mathcal{A}^{op}, z^{-1})$ defined by $\bar{f}\big((a_0, a_1, a_2)\big) = \big(f(a_0), f(a_1), f(a_2)\big)$ is an F-isomorphism. Consequently, we obtain an F-isomorphism $\tilde{f} := \bar{f}^{-1} \circ h : (\mathcal{A}, z) \xrightarrow{\sim} (\mathcal{A}, z^{-1})$ with $\tilde{f}\big((0,1,0)\big) = (0,0,1)$. For the last statement, the composition map $\psi \circ \tilde{f}$ gives an F-isomorphism from $(\mathcal{A}, z)$ onto $\mathcal{J}'$. $\square$

Remark. In 6.14, if $(\mathcal{A}, z) \cong (\mathcal{A}, z^{-1})$ over Z, then $(\overline{\mathcal{A}}, z) \cong (\overline{\mathcal{A}}, z^{-1})$ over $\overline{Z}$. Since $(\overline{\mathcal{A}}, z)$ is an Albert division algebra, this cannot happen by [17], p.204 (see 6.11). Hence we always have $(\mathcal{A}, z) \not\cong (\mathcal{A}, z^{-1})$ over Z though it may happen that $(\mathcal{A}, z) \cong (\mathcal{A}, z^{-1})$ over F. For example, this is the case if $(\mathcal{A}, z)$ is a Jordan torus of Albert type below.

We start to classify Jordan tori of Albert type. Let $J = \oplus_{\boldsymbol{\alpha} \in \Lambda} J_{\boldsymbol{\alpha}}$ be a Jordan torus of Albert type, i.e., the central closure $\overline{J}$ is an Albert algebra over $\overline{Z}$. Recall that an Albert algebra is a 27-dimensional central simple exceptional Jordan algebra of degree 3. By 6.7 and the fact that we have degree 3 and dimension 27, it follows that $\operatorname{supp} J = \Lambda$, $n \geq 3$, and there exists a basis $\{\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_n\}$ of Λ such that the central grading group Γ of J is given as

$$\Gamma = 3\mathbb{Z}\boldsymbol{\sigma}_1 + 3\mathbb{Z}\boldsymbol{\sigma}_2 + 3\mathbb{Z}\boldsymbol{\sigma}_3 + \mathbb{Z}\boldsymbol{\sigma}_4 + \cdots + \mathbb{Z}\boldsymbol{\sigma}_n.$$

Let $U = \oplus_{\boldsymbol{\alpha} \in \Delta} J_{\boldsymbol{\alpha}}$, where

$$\Delta := \mathbb{Z}\boldsymbol{\sigma}_1 + \mathbb{Z}\boldsymbol{\sigma}_2 + 3\mathbb{Z}\boldsymbol{\sigma}_3 + \mathbb{Z}\boldsymbol{\sigma}_4 + \cdots + \mathbb{Z}\boldsymbol{\sigma}_n.$$

We claim that $Z(U) = Z(J)$. Since $U \subset J$, we have $Z(U) \supset Z(J)$ and so the central grading group Δ_1 of U lies in between Δ and Γ, i.e., $\Delta > \Delta_1 > \Gamma$. If $\Gamma \neq \Delta_1 \neq \Delta$, we get $|\Delta/\Delta_1| = 3$ since $|\Delta/\Gamma| = 9$. Hence the grading group of the central closure $\overline{U}$ is $\Delta/\Delta_1 = \mathbb{Z}_3$. But by 3.9(iv), this cannot happen. Suppose then that $\Delta_1 = \Delta$. Then U is commutative and associative. We show, using the method of Lemma 2 in [8] p.420, that this cannot happen:

Suppose that U is commutative and associative. Since $\overline{J}$ is an Albert division algebra by 3.4.7, the commutative associative subalgebra $\overline{Z} \otimes_Z U$ becomes a subfield of $\overline{J}$, which is 9-dimensional since $|\Delta/\Gamma| = 9$. But this is impossible. Indeed, take an algebraic closure Ω containing $\overline{Z}$. Then, in $\overline{J}_\Omega$, we have $1 = e_1 + \cdots + e_9$ where e_i for $i = 1, \dots, 9$ are orthogonal idempotents. Hence, by Lemma 1 in [8] p.229, the degree of $\overline{J}_\Omega$ is ≥ 9, which is a contradiction since the degree of $\overline{J}_\Omega$ is equal to the degree of $\overline{J}$ which is 3 (see [8] p.223).

Thus we get $\Delta_1 = \Gamma$, i.e., $Z(U) = Z$. In particular, we have $\overline{U} = \overline{Z} \otimes_Z U \subset \overline{J}$. Note that $\overline{U}$ is a central subalgebra of the division algebra $\overline{J}$ which is 9-dimensional since $|\Delta/\Gamma| = 9$. So by the classification of finite dimensional central simple Jordan algebras, $\overline{U}$ is special (see [8] Corollary 2 p.204 and p.207). Hence, by 2.11, $\overline{U}$ has degree 3. So U is a special Jordan torus of central degree 3. Therefore, by 6.13, ch. $F \neq 3$ and U can be identified with $F_{\boldsymbol{\omega}}^+$ if $\omega \in F$ and with $H(E_{\boldsymbol{\omega}}, \sigma)$ otherwise, where $E = F(\sqrt{-3}) = F(\omega)$.

We first consider the case $U = F_{\boldsymbol{\omega}}^+$. Let x be an arbitrary nonzero element in $J_{\boldsymbol{\sigma}_3}$. Let u_i be an arbitrary nonzero element in $J_{\boldsymbol{\sigma}_i}$ for $i \neq 3$ and $u_3 := x^3 \in J_{3\boldsymbol{\sigma}_3}$. Then $F_{\boldsymbol{\omega}} = F_{\boldsymbol{\omega}}[u_1^{\pm 1}, \dots, u_n^{\pm 1}]$ is a Δ-torus with a $\langle \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, 3\boldsymbol{\sigma}_3, \boldsymbol{\sigma}_4, \dots, \boldsymbol{\sigma}_n \rangle$-grading. Let Tr be the generic trace of $\overline{J}$ and $U^\perp = \{y \in J \mid \operatorname{Tr}(Uy) = 0\}$. We claim that $x, x^2 \in U^\perp$. Since Tr is Z-linear and U is a free Z-module with basis $\{u_1^i u_2^j \mid i, j = 0, 1, 2\}$, it is enough to show that $\operatorname{Tr}\big((u_1^i u_2^j) x^k\big) = 0$ for all $i, j = 0, 1, 2$ and $k = 1, 2$. Since such $(u_1^i u_2^j) x^k$ are all homogeneous and their degrees are not contained in Γ, we get, by 6.7, $\operatorname{Tr}\big((u_1^i u_2^j) x^k\big) = 0$. Hence our claim is settled.

Since $x^3 = u_3 \in Z$ is invertible and $\operatorname{Tr}(F_{\boldsymbol{\omega}}) \subset Z$, we have shown the conditions (i)-(iv) for $\mathcal{J} = J$, $\mathcal{U} = F_{\boldsymbol{\omega}}^+$ and $z = u_3$ in 6.14. Therefore, by 6.14, J contains a subalgebra J' so that

$$\begin{array}{rl} \text{Case (I)} & \varphi : (F_{\boldsymbol{\omega}}, u_3) \xrightarrow{\sim} J' \quad \text{is a } Z\text{-isomorphism} \\ & \text{with } \varphi\,|_{F_{\boldsymbol{\omega}}} = \mathrm{id} \text{ and } \varphi\big((0,1,0)\big) = x, \quad \text{or} \\ \text{Case (II)} & \psi : (F_{\boldsymbol{\omega}}, u_3^{-1}) \xrightarrow{\sim} J' \quad \text{is a } Z\text{-isomorphism} \\ & \text{with } \psi\,|_{F_{\boldsymbol{\omega}}} = \mathrm{id} \text{ and } \psi\big((0,0,1)\big) = x. \end{array}$$

We give a $\langle \boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n \rangle$-grading to $(F_{\boldsymbol{\omega}}, u_3)$ so that $\mathbb{A}_t = (F_{\boldsymbol{\omega}}, u_3)$ is the Albert torus, i.e., $\mathbb{A}_t = \oplus_{\boldsymbol{\alpha} \in \Lambda} F t_{\boldsymbol{\alpha}}$, and $t_{\boldsymbol{\sigma}_i} = u_i$ for $i \neq 3$ and $t_{\boldsymbol{\sigma}_3} = (0, 1, 0)$ (see 6.8(2)).

Case (I): We have $\varphi(t_{\boldsymbol{\sigma}_i}) = u_i \in J_{\boldsymbol{\sigma}_i}$ for $i \neq 3$ and $\varphi(t_{\boldsymbol{\sigma}_3}) = x \in J_{\boldsymbol{\sigma}_3}$. Thus one gets the injective Z-homomorphism $\varphi : \mathbb{A}_t \longrightarrow J$ with $\varphi(F t_{\boldsymbol{\sigma}_i}) = J_{\boldsymbol{\sigma}_i}$ for all $i = 1, \dots, n$. Since $\mathbb{A}_t$ is of strong type, we have, for any $\boldsymbol{\alpha} = \alpha_1 \boldsymbol{\sigma}_1 + \cdots + \alpha_n \boldsymbol{\sigma}_n \in \Lambda$,

$$0 \neq \varphi\big(u_1^{\alpha_1} \cdot (u_2^{\alpha_2} \cdot (t_{\boldsymbol{\sigma}_3}^{\alpha_3} \cdot (u_4^{\alpha_4} \cdots u_n^{\alpha_n})\ldots)\big) \in J_{\boldsymbol{\alpha}},$$

and hence φ is surjective. Therefore, $J \cong \mathbb{A}_t$ over Z and $J \cong_\Lambda \mathbb{A}_t$.

Case (II): We first show that $F_{\boldsymbol{\omega}} = F_{\boldsymbol{\omega}}[u_1^{\pm1}, \dots, u_n^{\pm1}]$ satisfies the condition (v) in 6.14. Clearly the F-linear map f from $F_{\boldsymbol{\omega}}$ into $F_{\boldsymbol{\omega}}^{op}$ defined by

$$f(u_1^{\alpha_1} u_2^{\alpha_2} u_3^{\alpha_3} \cdots u_n^{\alpha_n}) = u_2^{\alpha_1} u_1^{\alpha_2} u_3^{\alpha_3} \cdots u_n^{\alpha_n}$$

(exchange the first two variables and leave alone the remaining variables) for all $\boldsymbol{\alpha} = \alpha_1\boldsymbol{\sigma}_1 + \alpha_2\boldsymbol{\sigma}_2 + 3\alpha_3\boldsymbol{\sigma}_3 + \alpha_4\boldsymbol{\sigma}_4 + \cdots + \alpha_n\boldsymbol{\sigma}_n \in \Delta$ is an F-algebra isomorphism (see 6.11). It is also clear that $f \circ \mathrm{Tr} = \mathrm{Tr} \circ f$ and $f(u_3) = u_3$. Hence the condition (v) in 6.14 is satisfied, and so there exists an F-isomorphism $\tilde{f} : \mathbb{A}_t = (F_{\boldsymbol{\omega}}, u_3) \xrightarrow{\sim} (F_{\boldsymbol{\omega}}, u_3^{-1})$ with $\tilde{f} \mid_{F_{\boldsymbol{\omega}}} = f$ and $\tilde{f}\big((0,1,0)\big) = (0,0,1)$. Thus, by 6.14, we get an injective F-homomorphism $\psi \circ \tilde{f} : \mathbb{A}_t \longrightarrow J$ with $\psi \circ \tilde{f}(Ft_{\boldsymbol{\sigma}_1}) = J_{\boldsymbol{\sigma}_2}$, $\psi \circ \tilde{f}(Ft_{\boldsymbol{\sigma}_2}) = J_{\boldsymbol{\sigma}_1}$ and $\psi \circ \tilde{f}(Ft_{\boldsymbol{\sigma}_i}) = J_{\boldsymbol{\sigma}_i}$ for all $i = 3, \dots, n$. Since $\mathbb{A}_t$ is of strong type, we have, for any $\boldsymbol{\alpha} = \alpha_1\boldsymbol{\sigma}_1 + \cdots + \alpha_n\boldsymbol{\sigma}_n \in \Lambda$,

$$0 \neq \psi \circ \tilde{f}\big(u_1^{\alpha_2} \cdot (u_2^{\alpha_1} \cdot (t_{\boldsymbol{\sigma}_3}^{\alpha_3} \cdot (u_4^{\alpha_4} \cdots u_n^{\alpha_n})...)\big) \in J_{\boldsymbol{\alpha}},$$

and hence $\psi \circ \tilde{f}$ is surjective. Therefore, we obtain $J \cong \mathbb{A}_t$ over F. Note that $\psi \circ \tilde{f}$ is not graded for the $\langle \boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_n \rangle$-grading of $\mathbb{A}_t$. However, if we give a $\langle \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_3 \dots, \boldsymbol{\sigma}_n \rangle$-grading to $\mathbb{A}_t$, $\psi \circ \tilde{f}$ is graded and so $J \cong_\Lambda \mathbb{A}_t$.

We finally consider the case $U = H(E_{\boldsymbol{\omega}}, \sigma)$ where $E = F(\omega)$. Let $J_E = E \otimes_F J$ be the Jordan torus over E. Let $\tau := \sigma_E \otimes \mathrm{id}$ be a σ_E-semilinear involution of J_E over F where σ_E is the nontrivial Galois automorphism of E over F. Then $U_E = E \otimes_F U = E_{\boldsymbol{\omega}}^+$ is an E-subalgebra of J_E. Also, by 4.9(2), we have $\tau \mid_{E_{\boldsymbol{\omega}}} = \sigma$ since $\boldsymbol{\omega} \notin M_n(F)$. In particular, we have

$$\tau(u_1u_2) = \sigma(u_1u_2) = u_2u_1. \tag{6.15}$$

Now, since J is exceptional, so is J_E. Hence the Jordan torus J_E over E must be of Albert type since the other two types are special. Since J_E contains the subalgebra $U_E = E_{\boldsymbol{\omega}}^+$, we can apply the previous argument for $0 \neq x \in J_{\boldsymbol{\sigma}_3} \subset E \otimes_F J_{\boldsymbol{\sigma}_3}$. Precisely, for $u_3 := x^3$, we have

$$\tilde{\mathbb{A}}_t := (E_{\boldsymbol{\omega}}, u_3) \quad \text{be the Albert torus over } E,$$

and there are two cases: for $t := (0,1,0) \in \tilde{\mathbb{A}}_t$, Case (I): $l : J_E \tilde{\longrightarrow} \tilde{\mathbb{A}}_t$ such that $l\mid_{U_E} = l\mid_{E_{\boldsymbol{\omega}}} = \mathrm{id}$ and $l(x) = t$.

Case (II): $l' : J_E \tilde{\longrightarrow} \tilde{\mathbb{A}}_t$ such that $l'(x) = t$, $l'(u_1) = u_2$, $l'(u_2) = u_1$ and $l'\mid_{U_E} = l'\mid_{E_{\boldsymbol{\omega}}}$ is an automorphism of the associative algebra $E_{\boldsymbol{\omega}}$.

Case (I): Let $\tilde{\tau}$ be the induced involution of $\tilde{\mathbb{A}}_t$ from τ via the isomorphism between J_E and $\tilde{\mathbb{A}}_t$, namely, $\tilde{\tau} = l \circ \tau \circ l^{-1}$. Then we have $\tilde{\tau}(u_i) = u_i$ for $i = 1, \dots, n$ and $\tilde{\tau}(t) = t$. Moreover, by 6.15, we have $\tilde{\tau}(u_1u_2) = u_2u_1$. Let tr be the generic trace of $\overline{E}_{\boldsymbol{\omega}}$. Since $\mathrm{tr}(u_1) = \mathrm{tr}(u_2) = \mathrm{tr}(u_1u_2) = 0$, we have $\overline{u_1} = -\frac{1}{2}u_1$, $\overline{u_2} = -\frac{1}{2}u_2$ and $\overline{u_1u_2} = -\frac{1}{2}u_1u_2$. Hence, by 6.1,

$$(*) \qquad \begin{aligned}(u_1u_2)\cdot t &= (u_1u_2)\cdot(0,1,0) = (0, -\frac{1}{2}u_1u_2, 0)\\ &= u_1\cdot(0,u_2,0) = -2u_1\cdot(u_2\cdot t).\end{aligned}$$

On the other hand,

$$(**) \qquad \begin{aligned}(\omega u_1u_2)\cdot t &= (u_2u_1)\cdot t = \tilde{\tau}(u_1u_2)\cdot\tilde{\tau}(t)\\ &= \tilde{\tau}\big((u_1u_2)\cdot t\big) = -2\tilde{\tau}\big(u_1\cdot(u_2\cdot t)\big) \quad \text{by } (*)\\ &= -2\tilde{\tau}(u_1)\cdot\big(\tilde{\tau}(u_2)\cdot\tilde{\tau}(t)\big) = -2u_1\cdot(u_2\cdot t),\end{aligned}$$

and so $(u_1u_2)\cdot t = (\omega u_1u_2)\cdot t = \omega(u_1u_2)\cdot t$, which is absurd since $(u_1u_2)\cdot t \neq 0$.

Case (II): Similarly, let $\tilde{\tau} := l' \circ \tau \circ l'^{-1}$ be the induced involution of $\tilde{\mathbb{A}}_t$. Then we have $\tilde{\tau}(u_1) = u_1$, $\tilde{\tau}(u_2) = u_2$, $\tilde{\tau}(t) = t$ and $\tilde{\tau}(u_1u_2) = u_2u_1$. Thus $(*)$ and $(**)$ also hold for this $\tilde{\tau}$. So we get a contradiction.

Consequently, the case $\omega \notin F$ cannot happen. Thus we have proven the following:

Theorem 6.16. *Let J be a Jordan n-torus of Albert type over F. Then $n \geq 3$, $\omega \in F$ and $J \cong_\Lambda \mathbb{A}_t$ for some toral grading. Conversely, $\mathbb{A}_t$ is a Jordan torus of Albert type.* □

Combined with Proposition 6.13, we get the following result which is used in [2] Proposition 2.17 p.15 to classify extended affine Lie algebras of type G_2:

Corollary 6.17. *Let J be a Jordan torus over F of central degree 3. Then* $\mathrm{ch}.F \neq 3$ *and*

$$J \cong_\Lambda \begin{cases} F_{\boldsymbol{\omega}}^{+} \text{ or } \mathbb{A}_t & \text{if } \omega \in F\\ H(E_{\boldsymbol{\omega}}, \sigma) & \text{otherwise}\end{cases}$$

where $E = F(\omega)$. *Conversely, the algebras* $F^+_{\boldsymbol{\omega}}$, $H(E_{\boldsymbol{\omega}}, \sigma)$ *and* $\mathbb{A}_t$ *are Jordan tori of central degree* 3. $\square$

§7 Summary

By 4.11, 5.5 and 6.16, we complete the classification of Jordan tori:

Theorem 7.1. *Let J be a Jordan n-torus over F. Then J is graded isomorphic to one of the four special Jordan tori*

$$F^+_{\boldsymbol{q}},\ H(F_{\boldsymbol{\varepsilon}}, *),\ H(E_{\boldsymbol{\xi}}, \sigma) \text{ and } J_{S(m)}(\{a_{\boldsymbol{\epsilon}}\}_{\boldsymbol{\epsilon}\in I}),$$

or to the Albert torus $\mathbb{A}_t$ if $n \geq 3$ and F contains a primitive 3*rd root of unity.* $\square$

Also, by 5.6, we have the following:

Corollary 7.2. *Let J be a Jordan n-torus over an algebraically closed field F. Then J is graded isomorphic to one of the three special Jordan tori*

$$F^+_{\boldsymbol{q}},\ H(F_{\boldsymbol{\varepsilon}}, *) \text{ and a standard Clifford torus } J_{S(m)},$$

or to the Albert torus $\mathbb{A}_t$ if $n \geq 3$ and $\operatorname{ch}. F \neq 3$. $\square$

Remark. Martinez and Zelmanov classified strongly prime $\mathbb{Z}$-graded Jordan algebras of a certain type in [12]. Our Jordan tori are strongly prime $\mathbb{Z}^n$-graded Jordan algebras of a very special type. The intersection of their algebras and Jordan tori consists of Jordan 1-tori, which are isomorphic to the algebra of Laurent polynomials $F[t, t^{-1}]$.

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